1 Timeline until end of semester

- Nov 16/17: next class (someone will substitute for me)
  - do "6.4, 6.5. Areas and applications of integrals" (5 questions)
  - take "chapter 6 quiz" (6 questions)

    All six questions are taken from homework assignments.
- [Thanksgiving Holidays]
  - take "Test on chapters 5 and 6" (10 questions, 60min)

    You have until Dec 1 to take this online test.
    (See next slide for what to expect.)
- Nov 30/Dec 1: last class, used for review.
  - final exam (comprehensive)

    Current plan:
    - in-class exam on official date (see course website)
    - plus online final exam (until Dec 6)

2 What to expect on Chapter 5&6 test

You have until Dec 1 to take this online test.

Password will be emailed.

As usual, you have 60 min for 10 questions.

- 4 problems on Chapter 5
  - exponential growth of cell culture
  - continuous interest (2 problems)
  - elasticity of demand
- 6 problems on Chapter 6
  - compute an antiderivative
  - compute an integral like $\int_1^5 \left(2x^3 - \frac{1}{x}\right) \mathrm{d}x$
  - area under, say, $y = \frac{3}{x} + \sqrt{x} + 1$ from $x = 1$ to $x = 4$
  - area of region enclosed by two curves
  - compute average value of a function
  - compute consumer's surplus

All of these are taken from the homework assignments.

3 Antiderivatives

$F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$.

Example 1. An antiderivative of x^2 is $\frac{1}{3}x^3$.

Other antiderivatives? $\frac{1}{3}x^3 + 7$ or $\frac{1}{3}x^3 + C$, where C is any constant

We write: $\int x^2 \mathrm{d}x = \dfrac{1}{3}x^3 + C$

Example 2. Find all antiderivatives of $5x^3$.

Solution.

$$\int x^3 \mathrm{d}x = \frac{1}{4}x^4 + C$$

$$\int 5x^3 \mathrm{d}x = \frac{5}{4}x^4 + C$$

Example 3. Find all antiderivatives of e^{2x}.

Solution. $\int e^{2x} \mathrm{d}x = \dfrac{1}{2}e^{2x} + C$

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Example 4. Find all antiderivatives of $2x^5 + 7 - \frac{3}{x}$.

Solution.

$$\int 2x^5 \mathrm{d}x = \frac{2}{6}x^6 + C = \frac{1}{3}x^6 + C$$

$$\int 7 \mathrm{d}x = 7x + C$$

$$\int \frac{1}{x} \mathrm{d}x = \ln|x| + C$$

[The $|...|$ allows x to be negative.]

$$\int \left(2x^5 + 7 - \frac{3}{x}\right) \mathrm{d}x = \frac{1}{3}x^6 + 7x - 3\ln|x| + C$$

Example 5. Suppose marginal cost is $\frac{3}{2}x^2 - 30x + 200$.

(a) Determine the cost function $C(x)$ if $C(0) = 4$.

(b) Find the additional cost when production is increased from 10 to 30 units.

Solution.

(a) Needed: $C(x)$ with $C'(x) = \frac{3}{2}x^2 - 30x + 200$ and $C(0) = 4$.

$C(x) = \frac{1}{2}x^3 - 15x^2 + 200x + \spadesuit$

[$\spadesuit$ (silly but YOLO!) because "C" is taken]

$C(0) = \spadesuit = 4$

Hence, $C(x) = \frac{1}{2}x^3 - 15x^2 + 200x + 4$.

(b) This is asking for $C(30) - C(10)$.

$C(x) = \frac{1}{2}x^3 - 15x^2 + 200x + \spadesuit$

$C(30) - C(10) = (6000 + \spadesuit) - (1000 + \spadesuit) = 5000$

Important. No need to know $\spadesuit = 4$ from the first part!
(Several homework assignments pick up on that point.)

4 Integrals

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$, where F is an antiderivative of f.

This is "the integral of $f(x)$ from $x=a$ to $x=b$".

It is common to write $\left[F(x)\right]_a^b = F(b) - F(a)$.

Example 6. Evaluate $\int_1^2 x^2 \mathrm{d}x$.

Solution. $\int_1^2 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_1^2 = \frac{2^3}{3} - \frac{1}{3} = \frac{7}{3}$

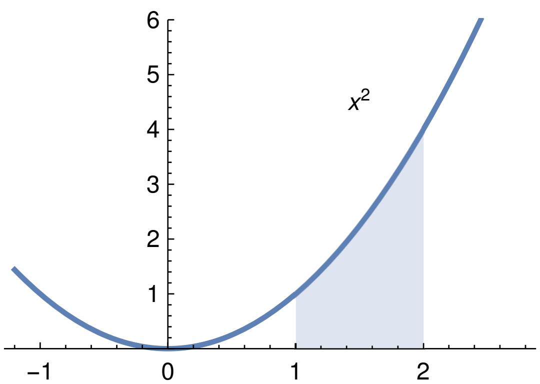
4.1 Applications

- $\int_a^b f(x)\mathrm{d}x$ is the **area** under the graph of $f(x)$ between a and b

  For instance. The previous example shows that the shaded area is

  $$\int_1^2 x^2 \mathrm{d}x = \frac{7}{3}$$

  units.



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- Next class (or online homework):

  $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\mathrm{d}x$ is the **average value** of $f(x)$ between a and b

- Next class (or online homework):

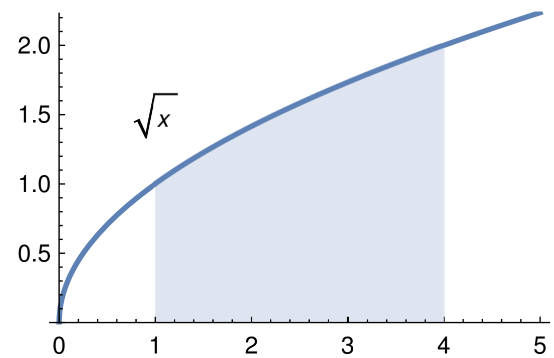
  The **consumer's surplus** at sales level A for a commodity with demand curve $p = f(x)$ is

$$\int_0^A [f(x) - f(A)]\mathrm{d}x.$$

Example 7. Determine the area under the curve $y = \sqrt{x}$ from $x = 1$ to $x = 4$.

Solution.

$$\begin{aligned}\int_1^4 \sqrt{x}\mathrm{d}x &= \left[\frac{2}{3}x^{3/2}\right]_1^4 \\ &= \frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2} \\ &= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}\end{aligned}$$



4.2 Basic properties

$$\int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x$$

$$\int_a^b [r f(x) + s g(x)]\mathrm{d}x = r\int_a^b f(x)\mathrm{d}x + s\int_a^b g(x)\mathrm{d}x$$

(Explored a little during homework!)

5 Preview: Applications of integrals (next class!)

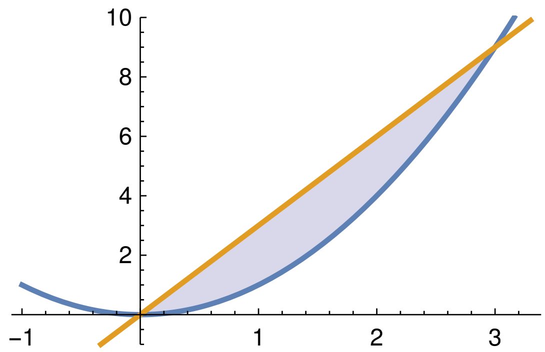
Example 8. Find the area of the region enclosed by the curves

$$y = x^2, \quad y = 3x.$$

Solution. First, make a sketch!

Doing so, we see that the area is:

$$\int_0^3 3x\,\mathrm{d}x - \int_0^3 x^2\,\mathrm{d}x = \frac{27}{2} - 9 = \frac{9}{2}$$



Example 9. Find the average value of the function $f(x) = x^2 + 1$ from $x = 0$ to $x = 3$.

Solution. $\frac{1}{3-0}\int_0^3 (x^2+1)\mathrm{d}x = \ldots = 4$

[Make a sketch of $f(x)$ and that the function increases from $f(0) = 1$ to $f(3) = 10$. If the graph was a line, the average value would be $\frac{1+10}{2} = 5.5$. Here, the average is less because the function "spends more time" at small values.]

Example 10. During a certain 24 hour period, the temperature at time t (measured in hours from the start of the period) was

$$T(t) = 43 + 9t - \frac{1}{2}t^2$$

degrees. What was the average temperature during that period?

Solution. The average temperature was

$\frac{1}{24-0}\int_0^{24}\left(43 + 9t - \frac{1}{2}t^2\right)\mathrm{d}t = \ldots = 55$ degrees.

Example 11. Find the consumer's surplus for the demand curve $p = 5 - \frac{x}{6}$ at the sales level $x = 12$.

Solution. Recall that the demand curve describes the relationship between price p and the quantity x of products that can be sold at that price ("demand").

The consumer's surplus at sales level A for a commodity with demand curve $p = f(x)$ is

$$\int_0^A [f(x) - f(A)]\mathrm{d}x.$$

Hence, here, consumer's surplus is

$$\int_0^{12} \left[\left(5 - \frac{x}{6}\right) - 3\right]\mathrm{d}x = 12.$$

What does consumer's surplus measure? If we want to sell A products at once ("in an open market"), the price needs to be set as $f(A)$ for a total revenue of $A \cdot f(A) = \int_0^A f(A)\mathrm{d}x$.

If a company wanted to get the most out of its customers, it could (in a non-open market) start by asking a very high price and selling just a few products, then lower the price a bit and so on.

Mathematically, the company would sell products in batches of Δx. Let $x_1, x_2, x_3, \ldots$ be the total number of products after selling $1, 2, 3, \ldots$ such batches (so $x_j = j \cdot \Delta x$). The amount of money paid by the consumers would be

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots$$

(This is probably easiest to see when looking at the graph of a specific demand curve.) It then becomes clear that, when taking smaller and smaller Δx, this amount of money approaches

$$\int_0^A f(x)\mathrm{d}x.$$

The consumer's surplus is the difference of this "extorted" amount and the amount in an open market:

$$\int_0^A f(x)\mathrm{d}x - A \cdot f(A) = \int_0^A [f(x) - f(A)]\mathrm{d}x$$

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Assignments.

- check out Sections 6.1, 6.2, 6.3 in the book
- do "6.1. Antiderivatives" (9 questions)
- do "6.2. Net change of functions" (6 questions)
- do "6.3. Areas under a graph" (4 questions)
- take "6.1, 6.2, 6.3. quiz on integrals" (6 questions)

  All six questions are taken from homework assignments.