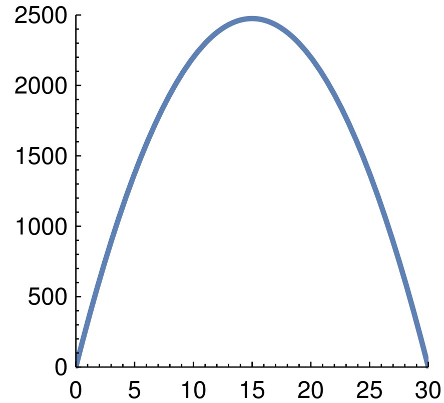
$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} \quad \left(\text{i.e. } \frac{\text{change in } y}{\text{change in } x}\right)$

$\boxed{f'(a) \text{ is the } \textbf{rate of change} \text{ of } f(x) \text{ at } x = a.}$

Example 1. Suppose $s(t)$ is the height in miles after t minutes of a rocket that is shot up vertically.



(a) What is the meaning of $s(5) = 1375$?

First off, units: $s(5)$ is miles.

After 5 minutes, the rocket is 1375 miles high.

(b) What is the meaning of $s'(5) = 220$?

First off, units: $s'(5)$ is miles/min.

After 5 minutes, the rocket has a **speed** of 220 miles/min (13200 miles/h).

(c) What is the meaning of $s''(5) = -22$?

First off, units: $s''(5)$ is (miles/min)/min, or miles/min^2.

After 5 minutes, the rocket has an **acceleration** of -22 miles/min^2.

Physics comment. Earth's gravitation is about 22 miles/min^2 (or 32.2 ft/sec^2).

In other words, our rocket is ballistic (only initially powered, then in free fall).

Just for fun. These numbers are all made up. However, they are (in some aspects) not too far off from the 2017/7/28 launch of a North Korea missile. That missile reached a height of about 2315 miles and landed after 47 minutes.

https://en.wikipedia.org/wiki/Hwasong-14

For comparison, the ISS is 205-270 miles above earth, the moon $238,900$ miles.

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1 Extrema

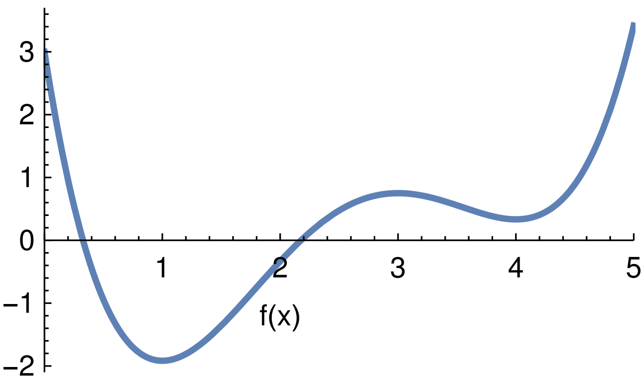
We will be very interested in extrema (maxima or minima):

- **(absolute) extrema**

  Here, the function is higher/lower than at any other point.

- **local extrema** (also called relative extrema)

  Here, the function is higher/lower than at nearby points.



Example 2. List all extrema of the above function $f(x)$.

(a) Local minima: at $x=1$ and at $x=4$

(b) Local maxima: at $x=3$

(c) Absolute minimum: at $x=1$

(d) Absolute maximum:

It does not look like $f(x)$ has an absolute maximum (instead the values for $x>5$ or $x<0$ seem to be growing without bound).

On the other hand, if the domain of $f(x)$ is only the interval $[0,5]$ (that is, $f(x)$ is only defined for $0\leqslant x\leqslant 5$), then the absolute maximum is at $x=5$.

(first-derivative rule)

- If $f'(a)>0$, then $f(x)$ is increasing at $x=a$.
- If $f'(a)<0$, then $f(x)$ is decreasing at $x=a$.
- If $f'(a)=0$, then $f(x)$ might have a relative extremum at $x=a$.

Such a (where $f'(a)=0$) are called **critical values**.

However, we need to investigate further these are indeed extrema.

(That's the point of the first- and second-derivative tests.)

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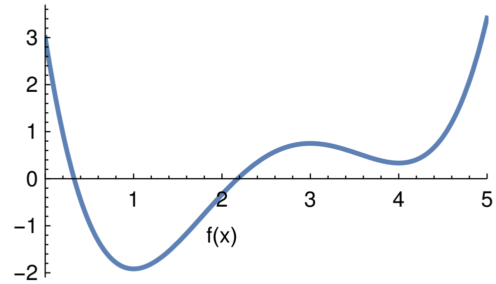
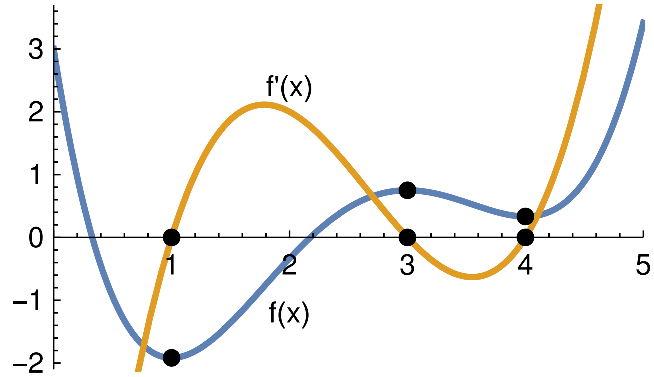
Example 3. On which intervals is $f(x)$ increasing/decreasing?

(a) $f(x)$ is increasing for: $1 < x < 3$ and $x > 4$

(b) $f(x)$ is decreasing for: $x < 1$ and $3 < x < 4$

(c) Sketch $f'(x)$.





2 Concavity

Example 4. For the same $f(x)$, describe the slopes between $x = 1$ and $x = 3$.

For $1 < x < 3$, the slopes are positive (i.e. $f(x)$ is increasing).

But we can say more:

The slopes are increasing from $x = 1$ until $x \approx 1.8$ (the maximal slope is about 2.1), then the slopes are descreasing from $x \approx 1.8$ to $x = 3$.

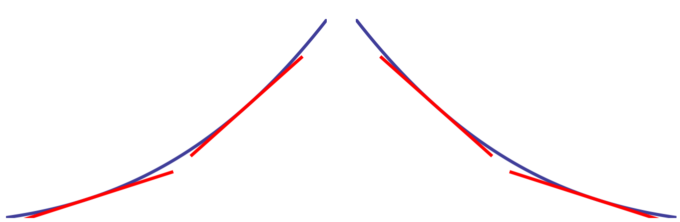
The point $x \approx 1.8$ is special. It is an **inflection point** (see below).

In any case, slopes are changing. It is of interest whether slopes are increasing or decreasing.

$f(x)$ is **concave up** (at $x = a$):

$\Longleftrightarrow$ graph lies above tangent line (around $x = a$)

$\Longleftrightarrow$ $f'(x)$ is increasing (at $x = a$) (i.e. slopes are increasing)



Being concave down is defined analogously.

Recall that derivatives can tell us whether a function is increasing!

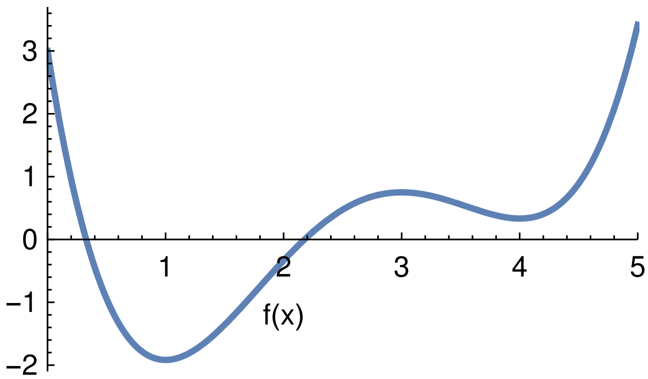
If $f''(a) > 0$, then $f'(x)$ is increasing at $x = a$.

(second-derivative rule)

- If $f''(a) > 0$, then $f(x)$ is concave up at $x = a$.
- If $f''(a) < 0$, then $f(x)$ is concave down at $x = a$.
- If $f''(a) = 0$, then $f(x)$ might have an inflection point at $x = a$.

An **inflection point** is a point, where concavity is changing.

[From concave up to down, or the other way around.]



Example 5. Approximately, on which intervals is $f(x)$ concave up/down?

(a) $f(x)$ is concave up for: $x < 1.8$ and $x > 3.5$

(b) $f(x)$ is concave down for: $1.8 < x < 3.5$

(c) $f(x)$ has inflection points at: $x \approx 1.8$ and $x \approx 3.5$

Here is a visual way to think of concavity and inflection points:

Imagine yourself riding a bike along the graph of $f(x)$. If the graph is a straight line, then you are steering neither left nor right. Usually, however, the graph is curved and you will have to steer either a little left or a little right.

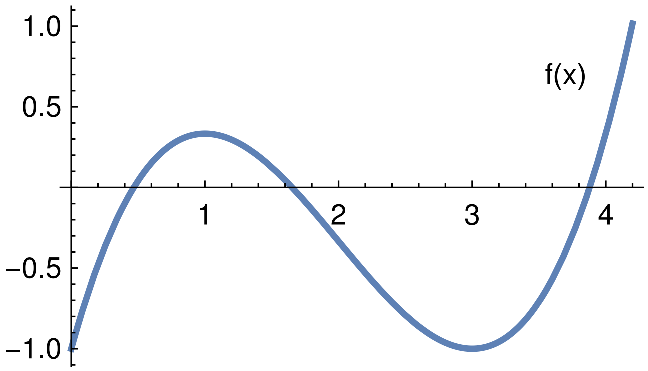
Steering left means the graph is concave up (at that point), steering right means the graph is concave down. An inflection point is a point where you are transitioning from steering one direction to the other.

Practice!

Even more so then usually, it is important to practice to get a firm feeling for all of these notions.

3 Finding local extrema

3.1 First-derivative test



Example 6. Approximately, describe $f(x)$. What are the implications for $f'(x)$ and $f''(x)$?

(a) increasing for: $x<1$ and $x>3$ $\quad f'>0$

(b) decreasing for: $1<x<3$ $\quad f'<0$

(c) local extrema: local max at $x=1$, local min at $x=3$ $\quad f'=0$

(d) concave up for: $x>2$ $\quad f''>0$

(e) concave down for: $x<2$ $\quad f''<0$

(f) inflection points: at $x\approx 2$ $\quad f''=0$

Recall. If $f(x)$ has a local extremum at $x=a$, then $f'(a)=0$

[or $f'(a)$ does not exist]

Why? Because $f'(a)>0$ would mean the function is increasing at $x=a$.

And, $f'(a)<0$ would mean the function is decreasing at $x=a$. In either case, there cannot be a local extremum at $x=a$.

To find extrema, we solve $f'(x)=0$ for x.

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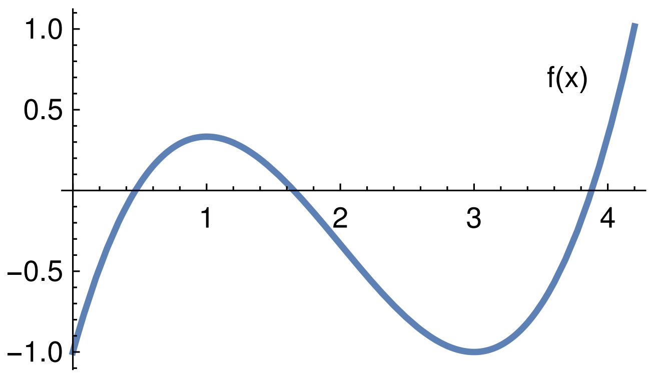
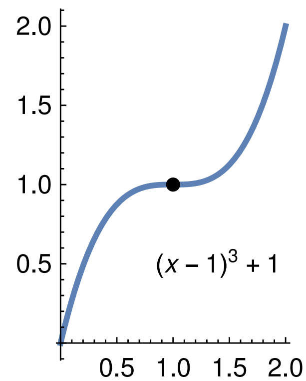
Such x are called **critical values**.

Not all critical values are extrema (see the plot of $(x-1)^3+1$ below).





Observation.

- At a local max, f changes from increasing to decreasing.
- At a local min, f changes from decreasing to increasing.

(first-derivative test) Suppose $f'(a)=0$.

- If $f'(x)$ changes from positive to negative at $x=a$, then $f(x)$ has a local max at $x=a$.
- If $f'(x)$ changes from negative to positive at $x=a$, then $f(x)$ has a local min at $x=a$.

Example 7. Find the local extrema of $f(x)=\frac{1}{3}x^3-2x^2+3x-1$.

Solution. We use the first-derivative test.

$f'(x)=x^2-4x+3$

Solving $f'(x)=0$ (that's a quadratic equation) we find $x=1$ and $x=3$.

intervals	$x<1$	$x=1$	$1<x<3$	$x=3$	$x>3$	
$f'(x)$	$+$	0	$-$	0	$+$	we can determine the sign by computing $f'(x)$ for some x in the interval
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$	

Hence, $f(x)$ has a local maximum at $x=1$.

And $f(x)$ has a local minimum at $x=3$.

[See Example 1 in Section 2.3 for more words.]

4 Next stop: pies!

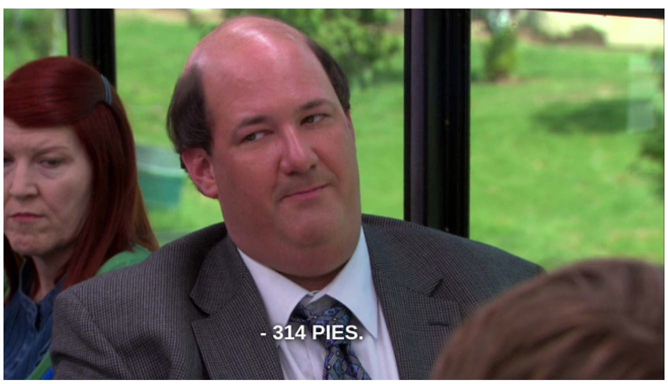
Angela: So, wait, when pies are involved, you can suddenly do math in your head?

Oscar: Hold on, Kevin, how much is 19,154 pies divided by 61 pies?

Kevin: 314 pies.

Oscar: What if it were salads?

Kevin: Well, it's the...carry the four...and...it doesn't work.



Any comments on Kevin's answer?

314 is a nice touch on the theme of "pi"es ($\pi = 3.14159...$).

However, the answer should have been just 314 (not 314 pies):

$$\frac{19154 \text{ pies}}{61 \text{ pies}} = 314$$

(Or, Oscar should have asked: "How much is 19,154 pies divided by 61?")

Moral. Watch your units!

As we have seen, this is super important for understanding rate of change, too.

The Office (Season 9, Episode 4):
http://www.simplethingcalledlife.com/stcl/when-pies-are-involved/

Assignments.

- check out Sections 2.1, 2.2 in the book
- do "2.1. Graphs" (5 questions)
- do "2.2. First and second derivative rules" (7 questions)
- have a first look at Section 2.3 in the book

  That's what we begin with next class.
- begin with "2.3. First and second derivative tests" (5 questions)
- I would suggest waiting to take "2.2, 2.3. Graphing quiz".

  These problems might feel a bit more tricky. More details next class.