Midterm #3

MATH 125 — Calculus 1
Tuesday, April 16

Please print your name:

Besides the allowed calculator, no notes or tools of any kind are permitted. There are 25 points in total.

Good luck!

Problem 1. (4 points) Compute the following indefinite integrals. [No need to show work here.]

(a) $\int [x^4 - 2x^2 + 7]\, \mathrm{d}x =$ $\boxed{}$

(b) $\int \left(\frac{1}{\sqrt{x}} + \frac{1}{x^3}\right) \mathrm{d}x =$ $\boxed{}$

(c) $\int \frac{1}{1+x^2}\, \mathrm{d}x =$ $\boxed{}$

(d) $\int [\sin(4x) + 3e^{2x}] \mathrm{d}x =$ $\boxed{}$

Solution.

(a) $\int [x^4 - 2x^2 + 7]\, \mathrm{d}x = \frac{x^5}{5} - \frac{2}{3}x^3 + 7x + C$

(b) $\int \left(\frac{1}{\sqrt{x}} + \frac{1}{x^3}\right) \mathrm{d}x = 2\sqrt{x} - \frac{1}{2x^2} + C$

(c) $\int \frac{1}{1+x^2}\, \mathrm{d}x = \arctan(x) + C$

(d) $\int [\sin(4x) + 3e^{2x}] \mathrm{d}x = -\frac{1}{4}\cos(4x) + \frac{3}{2}e^{2x} + C$ □

Problem 2. (2 points) Compute $\int_1^2 x^2 \mathrm{d}x$. [Show your work!]

Solution. $\int_1^2 x^2 \mathrm{d}x = \left[\frac{1}{3}x^3\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$ □

Problem 3. (2 points) Compute $\int_1^3 \frac{1}{x} \mathrm{d}x$. [Show your work!]

Solution. $\int_1^3 \frac{1}{x} \mathrm{d}x = \left[\ln(x)\right]_1^3 = \ln(3) - \ln(1) = \ln(3)$ □

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Problem 4. (2 points) Compute: $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{e^{2x}-e^{7x}}$ [Show your work!]

Solution. $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{e^{2x}-e^{7x}} \underset{\text{“}\frac{0}{0}\text{”}}{\overset{\text{LH}}{=}} \lim\limits_{x\to 0} \dfrac{3\cos(3x)}{2e^{2x}-7e^{7x}} = \dfrac{3\cdot 1}{2-7} = -\dfrac{3}{5}$ □

Problem 5. (2 points) Compute: $\lim\limits_{x\to 0^+} x^2 \ln(x)$. [Show your work!]

Solution. This limit is of the form “$0\cdot\infty$”. To apply L'Hospital, we first rewrite $x^2\ln(x) = \frac{\ln(x)}{x^{-2}}$.

$$\lim_{x\to 0^+} x^2\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}} \overset{\text{LH}}{=} \lim_{x\to 0^+} \frac{x^{-1}}{-2x^{-3}} = \lim_{x\to 0^+} -\frac{x^2}{2} = 0$$ □

Problem 6. (2 point) Compute: $\sum\limits_{k=2}^{4} \dfrac{(-2)^k}{k-1}$ [Show your work!]

Solution. $\sum\limits_{k=2}^{4} \dfrac{(-2)^k}{k-1} = \dfrac{4}{1} - \dfrac{8}{2} + \dfrac{16}{3} = \dfrac{16}{3}$ □

Problem 7. (3 points) Let A be the (net) area between the x-axis and $f(x)$ for x in $[1,5]$.

(a) Write down a Riemann sum for A using 3 intervals (of equal size) and midpoints.

(b) Using sigma notation, write down a Riemann sum for A using n intervals (of equal size) and midpoints.

Solution.

(a) Each interval has length $\frac{5-1}{3} = \frac{4}{3}$. The first interval is $\left[1, 1+\frac{4}{3}\right]$, which has midpoint $1+\frac{2}{3} = \frac{5}{3}$.

The 3 midpoints are $\frac{5}{3}$, $\frac{5}{3}+\frac{4}{3} = 3$, $3+\frac{4}{3} = \frac{13}{3}$ (each is $\frac{4}{3}$ after the previous).

The Riemann sum therefore is $\dfrac{4}{3}\left[f\left(\dfrac{5}{3}\right) + f(3) + f\left(\dfrac{13}{3}\right)\right]$.

(b) Each interval has length $\frac{5-1}{n} = \frac{4}{n}$. The first interval is $\left[1, 1+\frac{4}{n}\right]$, which has midpoint $1+\frac{2}{n}$.

The n midpoints are $1+\frac{2}{n}$, $1+\frac{2}{n}+\frac{4}{n}$, $1+\frac{2}{n}+2\cdot\frac{4}{n}$, ..., $1+\frac{2}{n}+(n-1)\cdot\frac{4}{n}$ (each is $\frac{4}{n}$ after the previous).

The Riemann sum therefore is $\dfrac{4}{n}\sum\limits_{k=0}^{n-1} f\left(1+\dfrac{2}{n}+k\cdot\dfrac{4}{n}\right)$. □

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Problem 8. (4 points) Suppose you have 100 m of fencing and want to fence off a rectangular field that borders a straight river (no fence is needed alongside the river). What is the maximum area you can fence off?

Solution. Let a and b be the lengths of the two sides (in m) of the field, with b the one along the river.

Then the required fencing is $2a+b$ and the area to be maximized is $A=a\cdot b$. Since $2a+b=100$ we find $b=100-2a$ and hence $A=a\cdot(100-2a)$. To find the maximum of A for a in $[0,50]$ we compute the critical points:

$$\frac{\mathrm{d}A}{\mathrm{d}a}=100-4a=0 \quad\Longrightarrow\quad a=25$$

Since the maximum clearly does not occur for the endpoints $a=0$ and $a=50$, the maximum must occur at the only critical point $a=25$. The corresponding maximum area is $25\cdot(100-2\cdot25)=1250\text{ m}^2$. □

Problem 9. (4 points)

(a) Estimate the average value of $f(x)=x^2$ on $[0,2]$ using a Riemann sum with 3 intervals and midpoints.

(b) Compute the (exact) average value of $f(x)=x^2$ on $[0,2]$.

Solution.

(a) Each interval has length $\frac{2-0}{3}=\frac{2}{3}$. The first interval is $\left[0,\frac{2}{3}\right]$ and has midpoint $\frac{1}{3}$.

The 3 midpoints therefore are $\frac{1}{3}$, $\frac{1}{3}+\frac{2}{3}=1$, $1+\frac{2}{3}=\frac{5}{3}$ (each is $\frac{2}{3}$ after the previous).

The estimate for the average value is

$$\frac{1}{2-0}\left(\frac{2}{3}f\left(\frac{1}{3}\right)+\frac{2}{3}f(1)+\frac{2}{3}f\left(\frac{5}{3}\right)\right)=\frac{1}{3}\left(f\left(\frac{1}{3}\right)+f(1)+f\left(\frac{5}{3}\right)\right)=\frac{1}{3}\left(\frac{1}{9}+1+\frac{25}{9}\right)=\frac{35}{27}.$$

(b) The average value is $\frac{1}{2-0}\int_0^2 x^2\mathrm{d}x=\frac{1}{2}\left[\frac{1}{3}x^3\right]_0^2=\frac{1}{2}\left(\frac{8}{3}-0\right)=\frac{4}{3}$.

[For comparison, our estimate was $\frac{35}{27}\approx1.296$.] □

(extra scratch paper)