Practice for Midterm #2

MATH 125 — Calculus 1
Tuesday, March 12

Please print your name:

Besides the allowed calculator, no notes or tools of any kind will be permitted.

- Have another look at the homework, especially those problems that you struggled with.
- Retake Quizzes 4, 5, and 6! (Versions with and without solutions are posted to our course website.)
- Go through the lecture sketches (posted to our course website) and do the problems we did in class (ignore the solutions until you have solved the problem yourself).

Problem 1. Compute the following derivatives.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}[2x^4 - 3\sqrt{x} + 7x - 4^2]$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^7\sin(4x+5)}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}e^{x^2-3}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}[x^3\ln(x^5+\cos(2x))]$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\left(x^2+\frac{1}{x}\right)$

(f) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+(x^2-1)\tan^{-1}(3x+2)}$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}[2x^4 - 3\sqrt{x} + 7x - 4^2] = 8x^3 - \frac{3}{2}x^{-1/2} + 7$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^7\sin(4x+5)} = -\frac{1}{(x^7\sin(4x+5))^2}\cdot(7x^6\sin(4x+5) + x^7\cos(4x+5)\cdot 4)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}e^{x^2-3} = 2xe^{x^2-3}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}[x^3\ln(x^5+\cos(2x))] = 3x^2\ln(x^5+\cos(2x)) + x^3\frac{5x^4 - \sin(2x)\cdot 2}{x^5+\cos(2x)}$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\left(x^2+\frac{1}{x}\right) = \frac{2x - \frac{1}{x^2}}{\sqrt{1-\left(x^2+\frac{1}{x}\right)^2}}$

(f) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+(x^2-1)\tan^{-1}(3x+2)} = \frac{2x\tan^{-1}(3x+2) + (x^2-1)\frac{3}{1+(3x+2)^2}}{2\sqrt{1+(x^2-1)\tan^{-1}(3x+2)}}$

□

Problem 2. Compute the derivatives of the following functions.

(a) $(\cos(x))^x$

(b) $x^{\cos(x)}$

Solution. In both cases, we apply logarithmic differentiation.

(a) Let $y = (\cos x)^x$. Then $\ln(y) = x \ln(\cos(x))$. Differentiating both sides, we obtain

$$\begin{aligned}\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln(\cos(x)) + x \cdot \frac{1}{\cos(x)} \cdot (-\sin(x)) \\ &= \ln(\cos(x)) - x\tan(x).\end{aligned}$$

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$, we find $\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x)^x[\ln(\cos x) - x\tan(x)]$.

(b) Let $y = x^{\cos(x)}$. Then $\ln(y) = \cos(x)\ln(x)$. Differentiating both sides, we obtain

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin(x)\ln(x) + \cos(x) \cdot \frac{1}{x}.$$

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives $\frac{\mathrm{d}y}{\mathrm{d}x} = x^{\cos(x)}\left[\frac{\cos(x)}{x} - \sin(x)\ln(x)\right]$. □

Problem 3. State the limit definition of $f'(3)$.

Solution. $f'(3) = \lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ □

Problem 4. The Lambert W function is defined as the inverse function of $f(x) = x e^x$. Determine $W'(x)$.

Solution. Note that $f'(x) = 1 \cdot e^x + x \cdot e^x = (x+1)e^x$.

Differentiating $x = f(W(x))$, we therefore obtain $1 = f'(W(x))W'(x)$, so that

$$W'(x) = \frac{1}{f'(W(x))} = \frac{1}{(W(x)+1)e^{W(x)}} = \frac{W(x)}{x(W(x)+1)}.$$

For the final equality, we simplified (optional!) using $W(x)e^{W(x)} = x$. □

Problem 5. Let $f(x) = \dfrac{1}{x^2 + g(-x)}$ and suppose that $g(-1) = 2$ and $g'(-1) = 3$. Find $f'(1)$.

Solution. Using the chain rule, $f'(x) = -\dfrac{2x + g'(-x) \cdot (-1)}{(x^2 + g(-x))^2}$. Hence, $f'(1) = -\dfrac{2-3}{(1+2)^2} = \dfrac{1}{9}$. □

Problem 6. Consider the curve $y^2\cos(x) + 2xy^3 = 4$.

(a) Using implicit differentiation, determine $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(b) Determine the line tangent to the curve at the point $(0, 2)$.

(c) Determine the line normal to the curve at the point $(0, 2)$.

Solution.

(a) Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $y^2\cos(x)+2xy^3=4$, we obtain $2y\cos(x)\frac{\mathrm{d}y}{\mathrm{d}x}-y^2\sin(x)+2y^3+6xy^2\frac{\mathrm{d}y}{\mathrm{d}x}=0$, so that

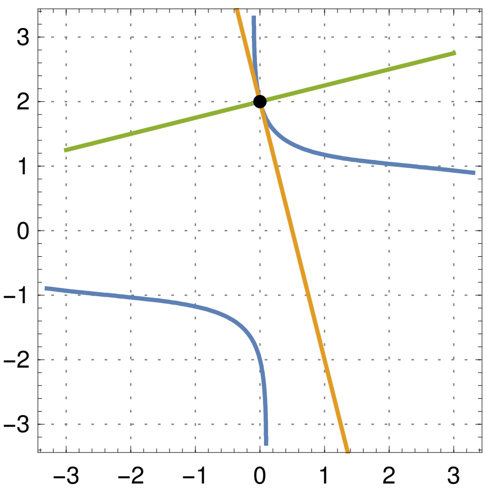
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2\sin(x)-2y^3}{2y\cos(x)+6xy^2}.$$

(b) The slope of the line tangent to the curve at $(0,2)$ is $\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{x=0,y=2}=\left[\frac{y^2\sin(x)-2y^3}{2y\cos(x)+6xy^2}\right]_{x=0,y=2}=\frac{0-16}{4+0}=-4.$

Hence, the tangent line has equation $y-2=-4(x-0)$, which simplifies to $y=-4x+2$.

(c) The normal line has slope $-\frac{1}{-4}=\frac{1}{4}$ and, thus, equation $y-2=\frac{1}{4}(x-0)$, which simplifies to $y=\frac{1}{4}x+2$.

Here is a sketch of the curve and the lines:



□

Problem 7. Consider the curve $xy+y^2=1$. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Solution. Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $xy+y^2=1$, we obtain $y+x\frac{\mathrm{d}y}{\mathrm{d}x}+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$,

so that $(x+2y)\frac{\mathrm{d}y}{\mathrm{d}x}=-y$ and, therefore, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-y}{x+2y}$. Consequently,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\frac{-y}{x+2y}=\frac{-\frac{\mathrm{d}y}{\mathrm{d}x}\cdot(x+2y)+y\cdot\left(1+2\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(x+2y)^2}=\frac{y+y\cdot\left(1-\frac{2y}{x+2y}\right)}{(x+2y)^2}=\frac{2y(x+y)}{(x+2y)^3}=\frac{2}{(x+2y)^3}.$$

In the final step, we simplified using $y(x+y)=xy+y^2=1$.

Comment. In these kinds of problems, alternative approaches can lead to solutions that look rather different (but, in the end, are equivalent). For instance, here, we could note that $x=\frac{1-y^2}{y}=\frac{1}{y}-y$. Hence, $\frac{\mathrm{d}x}{\mathrm{d}y}=-\frac{1}{y^2}-1$, so that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{-\frac{1}{y^2}-1}=-\frac{y^2}{y^2+1}$.

[Looks different than our earlier $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-y}{x+2y}$ but notice that, indeed, $\frac{-y}{x+2y}=\frac{-y}{\frac{1}{y}-y+2y}=\frac{-y^2}{1+y^2}$.]

By the chain rule, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{y^2}{y^2+1}\right]=\frac{\mathrm{d}}{\mathrm{d}y}\left[-\frac{y^2}{y^2+1}\right]\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2y}{(y^2+1)^2}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2y}{(y^2+1)^2}\frac{y^2}{y^2+1}=\frac{2y^3}{(y^2+1)^3}$. □

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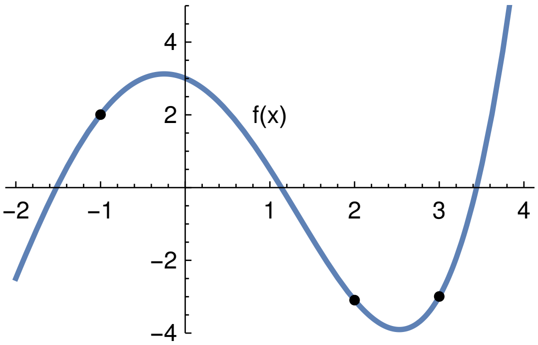
Problem 8. Use the graph below to fill in each entry of the grid with positive, negative or zero.



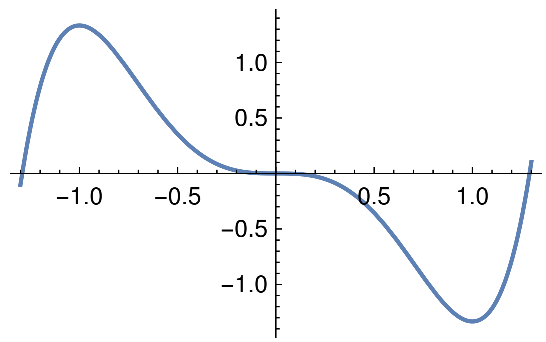
	$f(x)$	$f'(x)$	$f''(x)$
$x=-1$	$+$	$+$	$-$
$x=2$	$-$	$-$	$+$
$x=3$	$-$	$+$	$+$

Problem 9. The plot to the right shows the function $f(x)$.

(a) What are the critical points of $f(x)$?

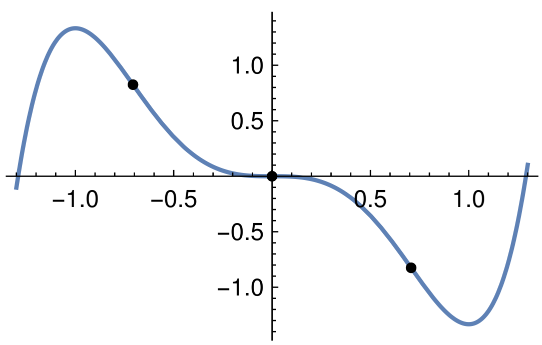
(b) Mark (roughly) the inflection points of $f(x)$ in the plot.

(c) $g(x)$ is a function such that $g'(x)=f(x)$. Does $g(x)$ have a local extremum at $x=0$?



Solution.

(a) The critical points of $f(x)$ are at $x=-1$, $x=0$ and $x=1$.

(b) 

(c) By the first-derivative test, $g(x)$ has a local extremum at $x=0$ if $g'(0)=0$ and $g'(x)$ changes sign at $x=0$.

Here, $g'(x)=f(x)$ changes from >0 to <0 at $x=0$, so that $g(x)$ has a local maximum at $x=0$. □

Problem 10. Roughly sketch a differentiable function $f(x)$ with the following property.

(a) $f'(0)=0$ but 0 is not a local extremum,

(b) $f'(0)>0$ and $f''(0)<0$.

Solution.

(a) You can sketch $f(x)=x^3$.

(b) You can sketch $f(x)=-(x-1)^2$ or any other function which is increasing at $x=0$ and concave down. □

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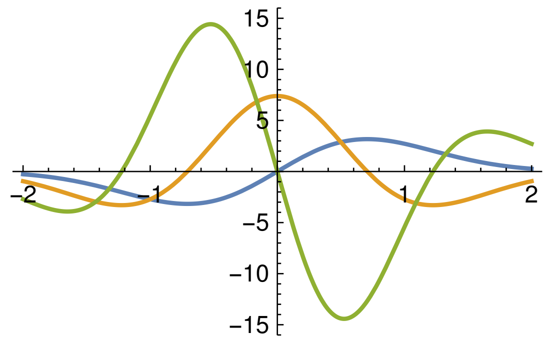
Problem 11. The plot to the right shows a function $f(x)$ as well as $f'(x)$ and $f''(x)$.

Which is which? Label the graphs accordingly.



Solution. The blue curve is $f(x)$, the orange curve is $f'(x)$, and green curve is $f''(x)$. □

Problem 12. The first and second derivatives of the function $f(x)$ have the following values:

	$x<-2$	$x=-2$	$-2<x<-1$	$x=-1$	$-1<x<0$	$x=0$	$0<x<1$	$x=1$	$1<x<3$	$x=3$	$x>3$
$f'(x)$	$-$	0	+	+	+	0	+	+	+	0	$-$
$f''(x)$	+	+	+	0	$-$	0	+	0	$-$	0	$-$

Determine the location of all local minima, local maxima and inflection points.

Solution. In summary, we have a local min at $x=-2$, a local max at $x=3$, and inflection points at $x=-1$, $x=0$, $x=1$.

The reasoning is as follows:

Local extrema can only occur when $f'(x)=0$. Hence, the candidates are $x=-2$, $x=0$ and $x=3$. If f' is changing from + to −, then we have a local max. Likewise, if f' is changing from − to +, then we have a local min.

- At $x=-2$: since f' is changing from − to +, there is a local min at $x=-2$.

  (Alternatively, we could have noticed that $f''(-2)>0$, which implies that this is a local min.)

- At $x=0$: since the sign of f' is not changing, we do not have a local extremum at $x=0$.

  (Since $f''(0)=0$, the second-derivative test would not help us decide whether this is a local extremum or not.)

- At $x=3$: since f' is changing from + to −, there is a local max at $x=3$.

  (Since $f''(0)=0$, the second-derivative test would not help us decide whether this is a local extremum or not.)

Inflection points can only occur when $f''(x)=0$. Hence, the candidates are $x=-1$, $x=0$, $x=1$ and $x=3$. Recall that $f(x)$ has an inflection point at $x=a$ if f'' is changing sign at $x=a$ (i.e. concavity is changing).

- At $x=-1$: since f'' is changing from + to −, there is an inflection point at $x=-1$.

- At $x=0$: since f'' is changing from − to +, there is an inflection point at $x=0$.

- At $x=1$: since f'' is changing from + to −, there is an inflection point at $x=1$.

- At $x=3$: since the sign of f'' is not changing (f is concave down before and after), we do not have an inflection point at $x=3$. □

Problem 13. Determine all local extrema of the function $f(x)=x^4-\frac{4}{3}x^3-4x^2+24x+1$. You may use that the critical points are at $x=-1$, $x=0$ and $x=2$.

Solution. Since the derivatives of $f(x)$ are pleasant to compute, we will use the second-derivative test.

$f'(x)=4x^3-4x^2-8x+24$

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$f''(x) = 12x^2 - 8x - 8$

Since $f''(-1) = 12 + 8 - 8 = 12 > 0$, $f(x)$ has a local min at $x = -1$.

Since $f''(0) = -8 < 0$, $f(x)$ has a local max at $x = 0$.

Since $f''(2) = 48 - 16 - 8 = 24 > 0$, $f(x)$ has a local min at $x = 2$.

Alternative. Since we have a complete list of critical points (i.e. there is no other x for which $f'(x) = 0$), we can also conveniently use the first-derivative test. However, since the second derivative is so easy to compute, the second-derivative test should be our first choice. □

Problem 14. Consider the function $s(t) = 2t^3 - 9t^2 + 12t$.

(a) What is the maximal value $s(t)$ attained on the interval $\left[0, \frac{1}{2}\right]$?

(b) Determine all local extrema of $s(t)$.

(c) On which intervals is $s(t)$ increasing?

(d) On which intervals is $s(t)$ concave up?

(e) Determine all inflection points of $s(t)$.

Solution.

(a) $s'(t) = 6t^2 - 18t + 12 = 6(t^2 - 3t + 2) = 6(t-1)(t-2)$

Hence, the critical points of $s(t)$ are at $t = 1$ and $t = 2$.

However, both are not in the interval $\left[0, \frac{1}{2}\right]$. Hence, the only candidates for the absolute maximum of $s(t)$ on $\left[0, \frac{1}{2}\right]$ is at the endpoints.

Since $s(0) = 0$ and $s\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{9}{4} + 6 = 4$, we conclude that the maximal value on $\left[0, \frac{1}{2}\right]$ is $s\left(\frac{1}{2}\right) = 4$.

(b) We already know that the local extrema can only occur at the critical values $t = 1$ and $t = 2$. Since the derivatives of $s(t)$ are so pleasant to compute, we will use the second-derivative test.

$s''(t) = 12t - 18$

$s''(1) = -6 < 0$ implies that $s(t)$ has a local max at 1.

$s''(2) = 6 > 0$ implies that $s(t)$ has a local min at 2.

(c) Note that a function is increasing before a local max and decreasing after (and, likewise, decreasing before a local min and increasing after). Since $t = 1$ and $t = 2$ are the only points where $s(t)$ can change from increasing to decreasing (and the other way around), we conclude that $s(t)$ is increasing on $(-\infty, 1)$ and $(2, \infty)$.

(d) $s(t)$ is concave up when $s''(t) > 0$. Since $s''(t) = 12t - 18 = 6(2t - 3)$ this is the case for t in $\left(\frac{3}{2}, \infty\right)$.

(e) By the previous part, $s(t)$ has an inflection at $t = \frac{3}{2}$. □

Problem 15. Consider $f(x) = (x^2 - 2)e^{2x}$.

(a) Determine all local and absolute extrema of $f(x)$ on the interval $[-3, 3]$.

(b) Determine the inflection points of $f(x)$.

Solution.

(a) $f'(x) = 2(x-1)(x+2)e^{2x}$

Solving $f'(x) = 0$, we find that the only critical points are at $x = -2$ and $x = 1$.

The extreme values can only occur at -2, 1 (critical points) or at -3, 3 (endpoints).

x	-3		-2		1		3
$f(x)$	$7e^{-6} \approx 0.01735$	$\nearrow$	$2e^{-4} \approx 0.03663$	$\searrow$	$-e^2 \approx -7.389$	$\nearrow$	$7e^6 \approx 2824$
$f'(x)$		$+$	0	$-$	0	$+$	

In conclusion:

- There is an absolute minimum of $-e^2$ at $x = 1$ and an absolute maximum of $7e^6$ at $x = 3$.
- There is an additional local minimum of $7e^{-6}$ at $x = -3$ and a local maximum of $2e^{-4}$ at $x = -2$.

Comment. We could also use the second-derivative test to determine that there is a local max at -2 (this follows from $f''(-2) = -6e^{-4} \approx -0.1099 < 0$) and a local min at $x = 1$ (this follows from $f''(1) = 6e^2 \approx 44.33 > 0$). However, since we are looking at all critical points (and need to check function values at the endpoints as well), it is less work to apply the first-derivative test.

(b) Recall that an inflection point is a local extremum of $f'(x)$.

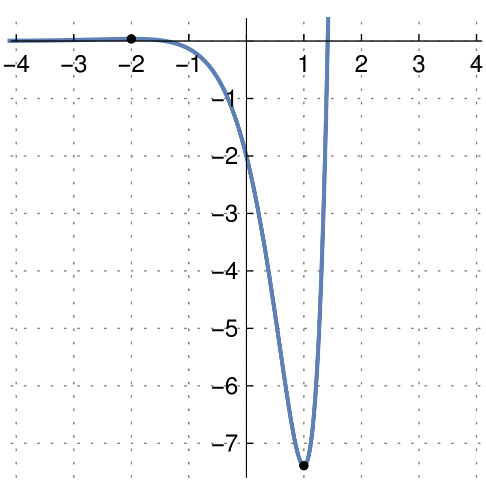
$f''(x) = 2(2x^2 + 4x - 3)e^{2x}$

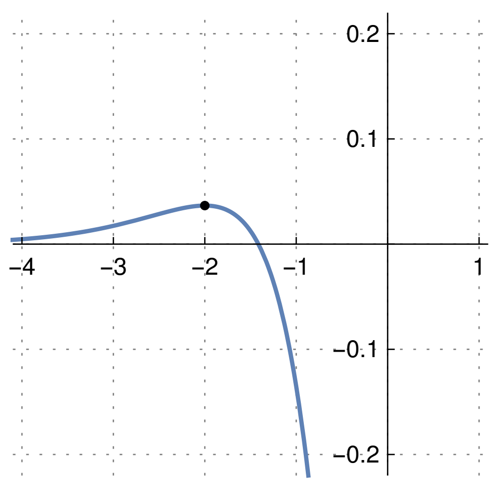
Solving $f''(x) = 0$, we find that $x = \frac{-2 \pm \sqrt{10}}{2}$ ($x \approx -2.581$ and $x \approx 0.581$). These are the critical points of $f'(x)$ and hence the only candidates for inflection points. We still need to check that concavity of $f(x)$ (i.e. the sign of $f''(x)$) is changing at these points. Since $f'' \neq 0$ for all other points, we can do so by checking the sign of $f''(x)$ at, say, $x = -3$, at $x = -2$ and $x = 1$. We find $f''(-3) = 6e^{-6} > 0$, $f''(-2) < 0$ (because $x = -2$ is a local max), and $f''(1) > 0$ (because $x = 1$ is a local min), so that the sign is indeed changing.

In conclusion, we have inflection points at $x = \frac{-2 \pm \sqrt{10}}{2}$.

Alternatively (advanced). There must be an inflection point between the local max at $x = -2$ (concave down) and the local min at $x = 1$ (concave up). Also, we can see that $f(x)$ has the x-axis as horizontal asymptote and that it approaches the x-axis from above as $x \to -\infty$. Hence, for very negative x, $f(x)$ must be concave up. Therefore, there must be a second inflection point before $x = -2$ (where $f(x)$ is concave down). Combined, there must be at least two inflection points, so that both candidates must indeed be inflection points.

A plot. Just for visual confirmation of our computations, here is a plot of $f(x)$. Note how, in the first plot, the situation at $x = -2$ is not really visible. The second plot addresses that point.





$\square$

Problem 16. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of an edge is 40 cm?

Solution. Let a be the edge length (in cm) of the cube.

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Then $V = a^3$ and $S = 6a^2$, so that $S = 6V^{2/3}$.

It follows that $\frac{\mathrm{d}S}{\mathrm{d}t} = 6 \cdot \frac{2}{3} \cdot V^{-1/3} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{4}{a} \frac{\mathrm{d}V}{\mathrm{d}t}$. When $a = 40$, we therefore have $\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{4}{40} \cdot 10 = 1\ \mathrm{cm}^2/\mathrm{min}$. □

Problem 17. Oil is leaking from a tanker and spreads in a circle whose area increases at a rate of 5 km^2/h. How fast is the radius of the spill increasing after 4 h?

Solution. Let A be the area (in km^2) and r the radius (in km) of the circular spill. Then A and r are related by the equation $A = \pi r^2$. It follows that the rates of change, with respect to time t (in h), are related by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t}.$$

We have $\frac{\mathrm{d}A}{\mathrm{d}t} = 5$. After $t = 4$, the area is $A = 4 \cdot 5$, so that the radius is $r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{20}{\pi}}$. It follows that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2\pi r} \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{5}{2\pi\sqrt{\frac{20}{\pi}}} = \frac{1}{4}\sqrt{\frac{5}{\pi}} \approx 0.3154\ \mathrm{km/h}.$$ □

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