The following rules are satisfied by definite integrals.

Think about each rule and make sure that it makes sense using the net area interpretation of the integral.

(1) $\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$

(2) $\int_a^a f(x)\mathrm{d}x = 0$

(3) $\int_a^b k f(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$

(4) $\int_a^b [f(x) \pm g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \pm \int_a^b g(x)\mathrm{d}x$

(5) $\int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x$

(6) $\text{min of } f \text{ on } [a,b] \leqslant \underbrace{\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x}_{\text{average of } f \text{ on } [a,b]} \leqslant \text{max of } f \text{ on } [a,b]$

(7) If $f(x) \geqslant g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\mathrm{d}x \geqslant \int_a^b g(x)\mathrm{d}x$.

Example 140. Suppose that

$$\int_1^3 f(x)\mathrm{d}x = 4, \quad \int_2^3 f(x)\mathrm{d}x = 5, \quad \int_1^3 g(x)\mathrm{d}x = 3.$$

(a) Determine $\int_1^2 f(x)\mathrm{d}x$.

(b) Determine $\int_1^3 [f(x) + g(x)]\mathrm{d}x$ and $\int_1^3 [7f(x) - g(x)]\mathrm{d}x$. What about $\int_1^3 f(x)g(x)\mathrm{d}x$?

(c) Determine $\int_3^1 \sqrt{5} f(x)\mathrm{d}x$.

Solution.

(a) $\int_1^2 f(x)\mathrm{d}x = \int_1^3 f(x)\mathrm{d}x - \int_2^3 f(x)\mathrm{d}x = 4 - 5 = -1$

(b) $\int_1^3 [f(x) + g(x)]\mathrm{d}x = \int_1^3 f(x)\mathrm{d}x + \int_1^3 g(x)\mathrm{d}x = 4 + 3 = 7$

$\int_1^3 [7f(x) - g(x)]\mathrm{d}x = 7\int_1^3 f(x)\mathrm{d}x - \int_1^3 g(x)\mathrm{d}x = 7 \cdot 4 - 3 = 25$

From the given information, we cannot determine $\int_1^3 f(x)g(x)\mathrm{d}x$. Explain why!

(c) $\int_3^1 \sqrt{5} f(x)\mathrm{d}x = -\sqrt{5}\int_1^3 f(x)\mathrm{d}x = -4\sqrt{5}$

Example 141. Compute $\int_0^2 \sqrt{4-x^2}\mathrm{d}x$.

Solution. Make a sketch! The area is the quarter of a circle with radius 2, which we know equals $\frac{1}{4}\cdot\pi\cdot 2^2=\pi$. Hence, $\int_0^2 \sqrt{4-x^2}\mathrm{d}x=\pi$.

Comment. Make sure to realize that it is not at all obvious what an antiderivative $F(x)$ of $f(x)=\sqrt{4-x^2}$ should be. In Calculus II you will learn techniques to find $F(x)=\frac{1}{2}x\sqrt{4-x^2}+2\arcsin(\frac{x}{2})$.

Of course, given $F(x)$, we can verify that $F'(x)=f(x)$. Do it! It is good review for the upcoming final exam.

Theorem 142. (Fundamental Theorem of Calculus, "first part") Let $f(x)$ be continuous on $[a,b]$. Then $F(x)=\int_a^x f(t)\mathrm{d}t$ is an antiderivative of f. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t=f(x).$$

What we saw earlier is often called the "second part".

Next time, we'll think about why the Fundamental Theorem of Calculus is true.

Example 143. Compute the following derivatives:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\int_2^x \frac{1}{t}\mathrm{d}t$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\int_2^{x^2} \frac{1}{t}\mathrm{d}t$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{t^2}\mathrm{d}t$ and $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^0 e^{t^2}\mathrm{d}t$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{2x^2}^{3x^2} e^{t^2}\mathrm{d}t$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \frac{1}{t}\mathrm{d}t=\frac{1}{x}$

Comment. Indeed, $\int_2^x \frac{1}{t}\mathrm{d}t=\ln(x)-\ln(2)$ which has derivative $\frac{1}{x}$.

However, the point is that we can determine the derivative of the integral without computing the integral itself (thus avoiding $\ln(x)$). This is crucial in the last two parts, because the antiderivative of e^{x^2} cannot be expressed in terms of functions we are familiar with.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\int_2^{x^2} \frac{1}{t}\mathrm{d}t=\frac{1}{x^2}\cdot 2x=\frac{2}{x}$

Step by step. Writing $f(x)=\int_1^x \frac{1}{t}\mathrm{d}t$, we are asked for $\frac{\mathrm{d}}{\mathrm{d}x}f(x^2)=f'(x^2)\cdot 2x$.

From the first part, we know that $f'(x)=\frac{1}{x}$.

For comparison. Again, we can (but shouldn't) compute $\int_2^{x^2} \frac{1}{t}\mathrm{d}t=\ln(x^2)-\ln(2)=2\ln(x)-\ln(2)$.

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{t^2}\mathrm{d}t=e^{x^2}$ and $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^0 e^{t^2}\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\left[-\int_0^x e^{t^2}\mathrm{d}t\right]=-e^{x^2}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{2x^2}^{3x^2} e^{t^2}\mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^{3x^2} e^{t^2}\mathrm{d}t-\int_0^{2x^2} e^{t^2}\mathrm{d}t\right]=e^{(3x^2)^2}\cdot 6x-e^{(2x^2)^2}\cdot 4x=6x e^{9x^4}-4x e^{4x^4}$