Example 138. (cont'd)

(a) Compute the (exact) average value of $f(x) = \frac{1}{\sqrt[3]{x}}$ on $[1, 3]$.

(b) Estimate the average value of $f(x) = \frac{1}{\sqrt[3]{x}}$ on $[1, 3]$ using a Riemann sum with 3 intervals and midpoints.

(c) Write down an estimate for the average value of $f(x) = \frac{1}{\sqrt[3]{x}}$ on $[1, 3]$ using a Riemann sum with n intervals and midpoints. (Use Σ-notation.)

Solution.

(a) The average value is $\frac{1}{3-1}\int_1^3 \frac{1}{\sqrt[3]{x}}\mathrm{d}x = \frac{1}{2}\left[\frac{3}{2}x^{2/3}\right]_1^3 = \frac{1}{2}\left(\frac{3}{2}\cdot 3^{2/3} - \frac{3}{2}\right) = \frac{3}{4}(3^{2/3} - 1)$.

(b) Each interval has length $\frac{3-1}{3} = \frac{2}{3}$. The first interval is $\left[1, 1 + \frac{2}{3}\right] = \left[1, \frac{5}{3}\right]$ and has midpoint $\frac{4}{3}$.

The 3 midpoints therefore are $\frac{4}{3}$, $\frac{4}{3} + \frac{2}{3} = 2$, $2 + \frac{2}{3} = \frac{8}{3}$ (each is $\frac{2}{3}$ after the previous).

The estimate for the average value is

$$\frac{1}{3-1}\left(\frac{2}{3}f\left(\frac{4}{3}\right) + \frac{2}{3}f(2) + \frac{2}{3}f\left(\frac{8}{3}\right)\right) = \frac{1}{3}\left(f\left(\frac{4}{3}\right) + f(2) + f\left(\frac{8}{3}\right)\right) = \frac{1}{3}\left(\sqrt[3]{\frac{3}{4}} + \frac{1}{\sqrt[3]{2}} + \sqrt[3]{\frac{3}{8}}\right) \approx 0.808.$$

[For comparison, the exact average is $\frac{3}{4}(3^{2/3} - 1) \approx 0.810$.]

(c) Each interval has length $\frac{3-1}{n} = \frac{2}{n}$. The first interval is $\left[1, 1 + \frac{2}{n}\right]$ and has midpoint $1 + \frac{1}{n}$.

The n midpoints therefore are $1 + \frac{1}{n}$, $1 + \frac{1}{n} + \frac{2}{n}$, $1 + \frac{1}{n} + 2\cdot\frac{2}{n}$, ..., $1 + \frac{1}{n} + (n-1)\cdot\frac{2}{n}$ (each is $\frac{2}{n}$ after the previous).

The estimate for the average value is:

$$\frac{1}{3-1}\sum_{k=0}^{n-1}\frac{2}{n}f\left(1 + \frac{1}{n} + k\cdot\frac{2}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(1 + \frac{1}{n} + k\cdot\frac{2}{n}\right)$$

[... average vs median ...]

Example 139. Let $f(x) = 6x^2 - 3x^3$.

(a) Compute $\int_0^2 f(x)\mathrm{d}x$.

(b) What is the average value of $f(x)$ for x in $[0, 2]$?

(c) What are the minimum and maximum value of $f(x)$ for x in $[0, 2]$?

Solution.

(a) $\int_0^2 f(x)\mathrm{d}x = \int_0^2 (6x^2 - 3x^3)\mathrm{d}x = \left[2x^3 - \frac{3}{4}x^4\right]_0^2 = \left(2\cdot 2^3 - \frac{3}{4}\cdot 16\right) - 0 = 4$

(b) The average value is $\frac{1}{2-0}\int_0^2 f(x)\mathrm{d}x = \frac{1}{2}\cdot 4 = 2$.

(c) Minimum value: $f(0) = f(2) = 0$

Maximum value: $f\left(\frac{4}{3}\right) = \frac{32}{9} \approx 3.55$ (Fill in all the details!)

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