Review. We write $\int f(x)\mathrm{d}x = F(x) + C$ if F is an antiderivative of f, meaning that $F' = f$.

Example 134. Determine the following indefinite integrals.

(a) $\int [x^4 + 3x + 7]\mathrm{d}x$

(b) $\int \sin(x)\mathrm{d}x$

(c) $\int \sin(3x)\mathrm{d}x$

(d) $\int \left[\frac{1}{x} + e^{2x} + e^{-x}\right]\mathrm{d}x$

(e) $\int \frac{3+\sqrt{t}}{t^5}\,\mathrm{d}t$

(f) $\int \frac{1}{x^2+1}\mathrm{d}x$

Solution.

(a) $\int [x^4 + 3x + 7]\mathrm{d}x = \frac{1}{4}x^4 + \frac{3}{2}x^2 + 7x + C$

(b) $\int \sin(x)\mathrm{d}x = -\cos(x) + C$

(c) $\int \sin(3x)\mathrm{d}x = -\frac{1}{3}\cos(3x) + C$

(d) $\int \left[\frac{1}{x} + e^{2x} + e^{-x}\right]\mathrm{d}x = \ln|x| + \frac{1}{2}e^{2x} - e^{-x} + C$

(e) $\int \frac{3+\sqrt{t}}{t^5}\,\mathrm{d}t = \int [3t^{-5} + t^{-9/2}]\mathrm{d}t = -\frac{3}{4}t^{-4} - \frac{2}{7}t^{-7/2} + C$

(f) $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan(x) + C$

Example 135. Determine $\int x^n\,\mathrm{d}x$ for all real numbers n.

Solution. By the power rule, $\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$ for all $n \neq -1$.

[Check that $n = 0$ works fine!]

On the other hand, for $n = -1$ we have $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$ (where either $x > 0$ or $x < 0$).

Recall. $\ln|x| = \begin{cases} \ln(x), & \text{if } x > 0, \\ \ln(-x) & \text{if } x < 0, \end{cases}$ and $\frac{\mathrm{d}}{\mathrm{d}x}\ln(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$.

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Example 136. Since $\frac{\mathrm{d}}{\mathrm{d}x} e^{x^2} = 2x e^{x^2}$, we find that $\int x e^{x^2} \mathrm{d}x = \frac{1}{2} e^{x^2} + C$.

Comment. You will learn about techniques to systematically find antiderivatives (like this one) in Calculus II. However, it should be noted that the situation for antiderivatives is much different than for derivatives:

While we can, in principle, take the derivative of any function given by a formula (built from addition, multiplication, division and composition using rational functions as well as exponential, logarithm and trig functions), the same is not true for antiderivatives.

For instance, it can be shown that $\int e^{x^2} \mathrm{d}x$, although it looks simpler than the one above, cannot be expressed via a formula using the functions we are familiar with.

Review.

- $\int_a^b f(x) \mathrm{d}x$ is the **net area** between the x-axis and $f(x)$ for x in $[a, b]$.
- $\frac{1}{b-a} \int_a^b f(x) \mathrm{d}x$ is the **average value** of $f(x)$ for x in $[a, b]$.
- $\int_a^b f(x) \mathrm{d}x = F(b) - F(a)$ where F is an antiderivative of f.

Example 137. Compute the (exact) average value of $f(x) = \frac{1}{\sqrt[3]{x}}$ on $[1, 3]$.

Solution. The average value is $\frac{1}{3-1} \int_1^3 \frac{1}{\sqrt[3]{x}} \mathrm{d}x = \frac{1}{2} \left[\frac{3}{2} x^{2/3} \right]_1^3 = \frac{1}{2} \left(\frac{3}{2} \cdot 3^{2/3} - \frac{3}{2} \right) = \frac{3}{4} (3^{2/3} - 1)$.