Example 127. Oil is leaking out of a tanker at an increasing rate:

time (h)	0	2	4	6	8
leakage (gal/h)	5	6	8	11	15

(a) Give an upper and lower estimate for the total quantity of oil leaked after 8 hours.

(b) Oil is continuing to leak at a rate of 15 gal/h after the first 8 h. If the tanker originally contained 200 gal of oil, how many more hours will elapse in the worst case before all oil has spilled? In the best case?

Solution.

(a) Lower estimate: we assume during the entire first 2 hours 5 gal/h were leaking, then 6 gal/h during the next 2 hours, ... Then the total amount leaked after 8 hours would be $2(5+6+8+11)=60\text{ gal}$.
Upper estimate: we assume during the entire first 2 hours 6 gal/h were leaking, then 8 gal/h during the next 2 hours, ... Then the total amount leaked after 8 hours would be $2(6+8+11+15)=80\text{ gal}$.

(b) Worst case: 80 gal leaked during the first 8 h. The remaining 120 gal would spill within the next $\frac{120}{15}=8\text{ h}$.
Best case: 60 gal leaked during the first 8 h. The remaining 140 gal would spill within the next $\frac{140}{15}\approx 9.33\text{ h}$ (that is, 9 h and 20 min).

Important comment. The sums in the first part are Riemann sums! Let $L(t)$ be the leakage (in gal/h) after time t (in h). Then $2(5+6+8+11)=2(L(0)+L(2)+L(4)+L(6))$ is the Riemann sum for the area between the x-axis and $L(t)$ for t in $[0,8]$ using 4 intervals (of width 2) and left endpoints.

Likewise, $2(6+8+11+15)=2(L(2)+L(4)+L(6)+L(8))$ is the Riemann sum for the same area using 4 intervals (of width 2) and right endpoints.

Indeed, we realize that the area between the x-axis and $L(t)$ for t in $[a,b]$ is the total amount leaked from $t=a$ to $t=b$.

Antiderivatives

Definition 128. F is an **antiderivative** for f if $F'=f$.

If F is an antiderivative for f, then all other antiderivatives are of the form $F(x)+C$.
We write

$$\int f(x)\mathrm{d}x = F(x)+C,$$

and refer to these antiderivatives as the **indefinite integral** of f.

Example 129. Find all antiderivatives for $f(x)=4-x^2$.

Solution. One antiderivative is $F(x)=4x-\frac{1}{3}x^3$. Hence, all antiderivatives are of the form $4x-\frac{1}{3}x^3+C$.
We write: $\int (4-x^2)\mathrm{d}x = 4x-\frac{1}{3}x^3+C$

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Example 130. Find all antiderivatives for $f(x)=\frac{1}{x}$ for $x>0$.

Solution. One antiderivative is $F(x)=\ln(x)$. Hence, all antiderivatives are of the form $\ln(x)+C$.

We write: $\int \frac{1}{x}\mathrm{d}x=\ln(x)+C$

Comment. When $x<0$, then $\frac{1}{x}$ has antiderivative $\ln(-x)$. Check that!

To accomodate both positive and negative x, one therefore frequently writes $\int \frac{1}{x}\mathrm{d}x=\ln|x|+C$. (Keeping in mind, that either $x>0$ or $x<0$.)

The integral

(Definite integrals)

$\int_a^b f(x)\mathrm{d}x$ is the **net area** between the x-axis and $f(x)$ for x in $[a,b]$.

Net area means that area above the x-axis is counted positively and area below the x-axis negatively.

More precisely, let $f(x)$ be continuous on $[a,b]$, and let S_n be any Riemann sum approximating the area using n intervals of equal size. We then define $\int_a^b f(x)\mathrm{d}x=\lim_{n\to\infty} S_n$ and it can be shown that the limit does not depend on the choice of Riemann sums (in fact, any sequence of Riemann sums, possibly using intervals of differing lengths, can be used, as long as the length of the largest subinterval approaches 0).

Theorem 131. (Fundamental Theorem of Calculus) Let $f(x)$ be continuous on $[a, b]$, and suppose that F is an antiderivative of f. Then

$$\int_a^b f(x)\mathrm{d}x=F(b)-F(a).$$

Recall that F is an antiderivative of f if it satisfies $F'(x)=f(x)$ for all x in $[a,b]$.

Example 132. Determine the area A between the x-axis and $f(x)=4-x^2$ for x in $[0,2]$.

Solution. $A=\int_0^2 (4-x^2)\mathrm{d}x$

If $F(x)=4x-\frac{1}{3}x^3$, then $F'(x)=4-x^2$.

Therefore, $A=\int_0^2 (4-x^2)\mathrm{d}x=F(2)-F(0)=\left(8-\frac{1}{3}\cdot 2^3\right)-0=\frac{16}{3}$.

For comparison. In Example 120, we estimated $A\approx 5.344$ via a Riemann sum using 8 intervals (of equal size) and midpoints. We now know that the true area is $A=\frac{16}{3}\approx 5.3333$.

(Averages)

The **average value** of $f(x)$ for x in $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$.

Example 133. Determine the average value m of $f(x)=\frac{1}{x}$ on $[1,3]$.

Solution. $m=\frac{1}{3-1}\int_1^3 \frac{1}{x}\mathrm{d}x$

If $F(x)=\ln(x)$, then $F'(x)=\frac{1}{x}$ on $[1,3]$.

Therefore, $m=\frac{1}{3-1}\int_1^3 \frac{1}{x}\mathrm{d}x=\frac{1}{2}(F(3)-F(1))=\frac{1}{2}(\ln(3)-\ln(1))=\frac{\ln(3)}{2}$.

For comparison. In Example 126, using a Riemann sum with 3 intervals and midpoints, we estimated $m\approx 0.5417$. We now know that the true average is $m=\frac{\ln(3)}{2}\approx 0.5493$.

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