Example 122. Evaluate $\sum_{k=2}^{5} 2^{-k}$.

Solution. $\sum_{k=2}^{5} 2^{-k} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{8+4+2+1}{32} = \frac{15}{32}$

Example 123. Write $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11}$ using sigma notation.

Solution. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} = \sum_{k=1}^{5} \frac{1}{2k+1}$ or, for instance, $\sum_{k=2}^{6} \frac{1}{2k-1}$ or $\sum_{k=0}^{4} \frac{1}{2k+3}$

Example 124. Write the following sums using sigma notation.

(a) $f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{7}{4}\right)$

(b) $f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{7}{4}\right) + f(2)$

(c) $f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right) + f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)$

Solution.

(a) $\sum_{k=0}^{7} f\left(\frac{k}{4}\right)$

Important note. $\frac{1}{4}\left[f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{7}{4}\right)\right]$ is the Riemann sum for the area between the x-axis and $f(x)$ for x in $[0, 2]$ using 8 intervals (of width $\frac{1}{4}$) and left endpoints.

(b) $\sum_{k=1}^{8} f\left(\frac{k}{4}\right)$

Important note. $\frac{1}{4}$ times this is the Riemann sum for the area between the x-axis and $f(x)$ for x in $[0, 2]$ using 8 intervals (of width $\frac{1}{4}$) and right endpoints.

(c) $\sum_{k=0}^{7} f\left(\frac{2k+1}{8}\right)$ or $\sum_{k=1}^{8} f\left(\frac{2k-1}{8}\right)$

Important note. $\frac{1}{4}$ times this is the Riemann sum for the area between the x-axis and $f(x)$ for x in $[0, 2]$ using 8 intervals (of width $\frac{1}{4}$) and midpoints.

Example 125. Let A be the area between the x-axis and $f(x)$ for x in $[1, 3]$.

(a) Write a Riemann sum for A using n intervals (of equal size) and midpoints.

(b) Evaluate the Riemann sum for $f(x) = \frac{1}{x}$ and $n = 3$.

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Solution.

(a) Each interval has length $\frac{3-1}{n}=\frac{2}{n}$, so that the midpoints are $\frac{2}{n}$ apart. The first interval is $\left[1,1+\frac{2}{n}\right]$ and has midpoint $1+\frac{1}{n}$. Hence the k-th interval has midpoint $1+\frac{1}{n}+(k-1)\frac{2}{n}=1+\frac{2k-1}{n}$. Hence, the requested Riemann sum is $\sum_{k=1}^{n}\frac{2}{n}f\left(1+\frac{2k-1}{n}\right)$.

(b) $\sum_{k=1}^{3}\frac{2}{3}f\left(1+\frac{2k-1}{3}\right)=\frac{2}{3}\left(f\left(\frac{4}{3}\right)+f\left(\frac{6}{3}\right)+f\left(\frac{8}{3}\right)\right)=\frac{2}{3}\left(\frac{3}{4}+\frac{3}{6}+\frac{3}{8}\right)=\frac{13}{12}\approx 1.0833$

Comment. The true area is $A=\ln(3)\approx 1.0986$.

(Averages)

The **average value** of $f(x)$ for x in $[a,b]$ is $\frac{1}{b-a}A$, where A is the area between the x-axis and $f(x)$ for x in $[a,b]$.

Soon. $\text{average of } f \text{ on } [a,b]=\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$

Example 126. Estimate the average value of $f(x)=\frac{1}{x}$ on $[1,3]$ using a Riemann sum with 3 intervals and midpoints.

Solution. The average is $\frac{1}{3-1}A=\frac{1}{2}A$, where A is the area between the x-axis and $f(x)=\frac{1}{x}$ for x in $[1,3]$. In the previous example, we estimated using a Riemann sum with 3 intervals and midpoints that $A\approx\frac{13}{12}$. Hence, our estimate for the average is $\frac{1}{2}\frac{13}{12}=\frac{13}{24}\approx 0.5417$.

Comment. The maximum of $f(x)=\frac{1}{x}$ on $[1,3]$ is $\frac{1}{1}=1$ and the minimum is $\frac{1}{3}$. The average had to be somewhere inbetween.

Comment. Note that $\frac{1}{2}$ times the Riemann sum is $\frac{1}{3}\left(f\left(\frac{4}{3}\right)+f\left(\frac{6}{3}\right)+f\left(\frac{8}{3}\right)\right)$, which we recognize as precisely the average of three of the function values.

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