Let A be the area between the x-axis and $f(x)$ for x in $[a,b]$.

Soon. $A = \int_a^b f(x)\mathrm{d}x$

Sums approximating the area A as in Example 118 are called **Riemann sums**:

(Riemann sums) In the most general case, we divide $[a,b]$ according to $a = x_0 < x_1 < \cdots < x_m = b$. In each of the m subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, we select a point c_i.

$$\text{Riemann sum} = \underbrace{(x_1 - x_0) f(c_1)}_{\substack{\text{rectangle \#1} \\ \text{width}=x_1-x_0 \\ \text{height}=f(c_1)}} + \underbrace{(x_2 - x_1) f(c_2)}_{\substack{\text{rectangle \#2} \\ \text{width}=x_2-x_1 \\ \text{height}=f(c_2)}} + \ldots + \underbrace{(x_m - x_{m-1}) f(c_m)}_{\substack{\text{rectangle \#}m \\ \text{width}=x_m-x_{m-1} \\ \text{height}=f(c_m)}}$$

Often, we choose the subintervals to be of the same length, say, Δx.
Typical natural choices for c_i in $[x_{i-1}, x_i]$ are:

- $c_i = x_{i-1}$ (left endpoint),
- $c_i = x_i$ (right endpoint),
- $c_i = \frac{x_{i-1} + x_i}{2}$ (midpoint),
- c_i equal to where f is maximal on $[x_{i-1}, x_i]$ (resulting in an "upper Riemann sum"),
- c_i equal to where f is minimal on $[x_{i-1}, x_i]$ (resulting in a "lower Riemann sum").

Example 119. Sketch the rectangles associated with the Riemann sum for the area between the x-axis and $f(x) = 4 - x^2$ for x in $[0,2]$ using 4 intervals (of equal size) and

(a) left endpoints,

(b) right endpoints,

(c) midpoints.

Solution. Do it! You can also look at these in the GeoGebra app we explored in class:
https://www.geogebra.org/m/zwsgarcp

Example 120. Again, let A be the area between the x-axis and $f(x) = 4 - x^2$ for x in $[0,2]$. Write down a Riemann sum for A using 8 intervals (of equal size) and

(a) left endpoints,

(b) right endpoints,

(c) midpoints.

Solution.

(a) The Riemann sum is $\frac{1}{4}\left[f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{7}{4}\right)\right] = \frac{93}{16} \approx 5.813$.
Since this is also the "upper sum", we have $A < 5.813$.

(b) The Riemann sum is $\frac{1}{4}\left[f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{7}{4}\right) + f(2)\right] = \frac{77}{16} \approx 4.813$.
Since this is also the "lower sum", we have $A > 4.813$.

(c) The Riemann sum is $\frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right) + f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right] = \frac{171}{32} \approx 5.344$. This is a very good estimate for A.

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As we can see from the previous example, it is desirable to have a better way to write sums with lots of terms.

(Sigma notation)

$$\sum_{k=r}^{s} a_k = a_r + a_{r+1} + \ldots + a_{s-1} + a_s$$

For instance, $\sum_{k=2}^{6} a_k = a_2 + a_3 + a_4 + a_5 + a_6$.

Example 121. Evaluate $\sum_{k=2}^{4} \frac{k}{k+1}$.

Solution. $\sum_{k=2}^{4} \frac{k}{k+1} = \frac{2}{3} + \frac{3}{4} + \frac{4}{5} = \frac{40+45+48}{60} = \frac{133}{60}$