# Sketch of Lecture 28

Fri, 3/29/2019

**(L'Hospital's Rule)** Suppose that $f(a) = g(a) = 0$ (i.e. $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form "$\frac{0}{0}$"). Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

[Fine print: around $a$, the functions $f, g$ need to be differentiable, and $g'(x) \neq 0$ for $x \neq a$.]

The same conclusion applies when $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form "$\frac{\infty}{\infty}$". Also, $a$ may be infinite.

**Careful!** L'Hospital's Rule can only be applied to indeterminate limits of the forms "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$". Applying it to other limits will give incorrect results.

**Why does L'Hospital's Rule work?** If $f(a) = g(a) = 0$ then, taking linear approximations,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a) \cdot (x - a)}{g(a) + g'(a) \cdot (x - a)} = \frac{f'(a)}{g'(a)}.$$

**More.** Other indeterminate cases include:

- "$0 \cdot \infty$" which can always be written as, say, "$\frac{0}{0} = \frac{0}{\frac{1}{\infty}}$".
- "$\infty - \infty$" which can sometimes be rewritten as "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" (for instance, by putting things on a common fraction).
- "$1^{\infty}$", "$0^0$" and "$\infty^0$" which can often be handled by appling $\ln$ to the limit in question.

**Example 108.** Determine $\lim_{x \to 0} \frac{\sin(5x)}{3x}$.

**Solution. (old)** We have seen this kind of limit earlier. It follows from $\underbrace{\lim_{x \to 0} \frac{\sin(x)}{x}}_{= \sin'(0)} = 1$ that $\lim_{x \to 0} \frac{\sin(5x)}{3x} = \frac{5}{3}$.

**Solution. (new)** This limit is of the form "$\frac{0}{0}$". We may therefore apply L'Hospital:

$$\lim_{x \to 0} \frac{\sin(5x)}{3x} \overset{\text{LH}}{=} \lim_{x \to 0} \frac{5\cos(5x)}{3} = \frac{5}{3}\cos(0) = \frac{5}{3}$$

**Example 109.** Determine $\lim_{x \to \infty} \frac{3x^2 - 7}{4x^2 + x + 2}$.

**Solution. (old)** We have seen this kind of limit earlier:

$$\lim_{x \to \infty} \frac{3x^2 - 7}{4x^2 + x + 2} = \lim_{x \to \infty} \frac{3 - \frac{7}{x^2}}{4 + \frac{1}{x} + \frac{2}{x^2}} = \frac{3 - 0}{4 + 0 + 0} = \frac{3}{4}$$

**Solution. (new)** This limit is of the form "$\frac{\infty}{\infty}$". We may therefore apply L'Hospital (twice):

$$\lim_{x \to \infty} \frac{3x^2 - 7}{4x^2 + x + 2} \overset{\text{LH}}{=} \lim_{x \to \infty} \frac{6x}{8x + 1} \overset{\text{LH}}{=} \lim_{x \to \infty} \frac{6}{8} = \frac{3}{4}$$

**Example 110.** Determine $\lim_{x \to \infty} \frac{x}{\ln(x)}$.

**Solution.** $\lim_{x \to \infty} \frac{x}{\ln(x)} \overset{\text{LH}}{=} \lim_{x \to \infty} \frac{1}{1/x} = \lim_{x \to \infty} x = \infty$

Armin Straub
straub@southalabama.edu

**Example 111.** Determine $\lim_{x\to\infty}\frac{\sin(x)}{\ln(x)}$.

**Solution.** Note that we cannot apply L'Hospital here because the limit is not of the forms "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" (the denominator goes to $\infty$ but the numerator oscillates between $-1$ and $1$). However, we can apply the sandwich theorem instead, using that $-\frac{1}{\ln(x)}\leqslant\frac{\sin(x)}{\ln(x)}\leqslant\frac{1}{\ln(x)}$ (for all $x>0$).

Since $\lim_{x\to\infty}-\frac{1}{\ln(x)}=0=\lim_{x\to\infty}\frac{1}{\ln(x)}$, the sandwich theorem implies that $\lim_{x\to\infty}\frac{\sin(x)}{\ln(x)}=0$.

**Example 112.** **(extra)** Determine $\lim_{x\to0}\frac{1-\cos(3x)}{x^2}$ and $\lim_{x\to0}\frac{1-\cos(3x)}{x^2+x}$.

**Solution.** Both limits are of the form "$\frac{0}{0}$".

$\lim_{x\to0}\frac{1-\cos(3x)}{x^2}\overset{\text{LH}}{=}\lim_{x\to0}\frac{3\sin(3x)}{2x}\overset{\text{LH}}{=}\lim_{x\to0}\frac{9\cos(3x)}{2}=\frac{9}{2}$ (the second limit is of the form "$\frac{0}{0}$" as well)

$\lim_{x\to0}\frac{1-\cos(3x)}{x^2+x}\overset{\text{LH}}{=}\lim_{x\to0}\frac{3\sin(3x)}{2x+1}=\frac{0}{1}=0$ (be careful not to apply L'Hospital again!)

**Example 113.** Determine $\lim_{x\to0^+}x^2\ln(x)$.

**Solution.** This limit is of the form "$0\cdot\infty$". To apply L'Hospital, we first rewrite $x^2\ln(x)=\frac{\ln(x)}{x^{-2}}$.

$$\lim_{x\to0^+}x^2\ln(x)=\lim_{x\to0^+}\frac{\ln(x)}{x^{-2}}\overset{\text{LH}}{=}\lim_{x\to0^+}\frac{x^{-1}}{-2x^{-3}}=\lim_{x\to0^+}-\frac{x^2}{2}=0$$

**Example 114.** Determine $\lim_{x\to0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]$.

**Solution.** This limit is of the form "$\infty-\infty$". To apply L'Hospital, we first rewrite $\frac{1}{x}-\frac{1}{e^x-1}=\frac{e^x-1-x}{x(e^x-1)}$ to get a limit of the form "$\frac{0}{0}$".

$$\lim_{x\to0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right]=\lim_{x\to0}\frac{e^x-1-x}{xe^x-x}\overset{\text{LH}}{=}\lim_{x\to0}\frac{e^x-1}{xe^x+e^x-1}\overset{\text{LH}}{=}\lim_{x\to0}\frac{e^x}{xe^x+2e^x}=\frac{e^0}{0+2e^0}=\frac{1}{2}$$

[Make sure to check that we can L'Hospital a second time because that limit is of the form "$\frac{0}{0}$" as well.]

| If $L=\lim_{x\to a}\ln(f(x))$, then $\lim_{x\to a}f(x)=\lim_{x\to a}e^{\ln(f(x))}=e^L$. |
|---|

**Example 115.** Determine $\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x$.

**Solution.** This limit is of the form "$1^\infty$". To apply L'Hospital, we first note that

$$\ln\left(\left(1-\frac{2}{x}\right)^x\right)=x\ln\left(1-\frac{2}{x}\right)=\frac{\ln\left(1-\frac{2}{x}\right)}{x^{-1}}$$

and that, as $x\to\infty$, the RHS results in a limit of the form "$\frac{0}{0}$".

$$L=\lim_{x\to\infty}\ln\left(\left(1-\frac{2}{x}\right)^x\right)=\lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}\right)}{x^{-1}}\overset{\text{LH}}{=}\lim_{x\to\infty}\frac{\frac{2x^{-2}}{1-\frac{2}{x}}}{-x^{-2}}=\lim_{x\to\infty}-\frac{2}{1-\frac{2}{x}}=-\frac{2}{1-0}=-2$$

As in the boxed formula above, we conclude that $\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x=e^L=e^{-2}$.

Armin Straub
straub@southalabama.edu