The mean value theorem

Theorem 99. (mean value theorem) Suppose $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Then there is (at least) one point c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Why? Note that the right-hand side is the average rate of change of f over the interval $[a,b]$. The mean value theorem says that the "instanteneous" rate of change $f'(c)$ must equal that average at some point c.

That's certainly plausible: if f' is continuous and $f'(c)$ never equals the average, then $f'(c)$ must be larger (or smaller) than that average for all c in (a,b). But that cannot be.

Rolle's Theorem. The special case when $f(a) = f(b) = 0$ is known as **Rolle's Theorem**. In that case, the conclusion is that there is (at least) one point c in (a,b) such that $f'(c) = 0$.

Note that Rolle's Theorem is saying that between two zeros of f, there must be a zero of f'.

Note. The conditions for the mean value theorem are so that it applies, for instance, to $f(x) = \sqrt{x}$, which is continuous on $[0,1]$ and differentiable on $(0,1)$ (but not differentiable at $x = 0$).

Example 100. Consider $f(x) = \frac{1}{x}$. Apply the mean value theorem to the interval $[1,2]$. Find the value(s) of c in $(1,2)$ that satisfy $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Solution. We have $f(1) = 1$ and $f(2) = \frac{1}{2}$. The mean value theorem, applied with $a = 1$ and $b = 2$, claims that there is (at least) one point c in $(1,2)$ such that

$$f'(c) = \frac{f(2) - f(1)}{2 - 1} = -\frac{1}{2}.$$

Since $f'(x) = -\frac{1}{x^2}$, we solve $f'(c) = -\frac{1}{2}$ to find that such a point c. The equation $-\frac{1}{c^2} = -\frac{1}{2}$ has the two solutions $c = \pm\sqrt{2}$, of which only $c = \sqrt{2}$ lies in the interval $(1,2)$.

Example 101. Show that $f(x) = x^4 - 5x + 1$ has exactly one zero in the interval $[0,1]$.

Solution. Note that $f(0) = 1$ and $f(1) = -3$. Since 0 is between these two values, the intermediate value theorem shows that there must be at least one c in $(0,1)$ such that $f(c) = 0$.

If there was a second zero of f, then, by Rolle's Theorem, the derivative $f'(x) = 4x^3 - 5$ must have a zero between the two zeros of f. However, $f'(x) < 0$ (and hence $f'(x) \neq 0$) for all x in $[0,1]$.

In conclusion, f must have exactly one zero in $[0,1]$.

Comment. Make a plot of $f(x)$ to graphically confirm our finding. With some more work, we could show that $f(x)$ has a total of two real zeros, the second of which is in $[1,2]$.

If $f'(x) = 0$ for all x in some interval, then f is constant on that interval.

Proof. If f was not constant on that interval, then $f(b) \neq f(a)$ for some a and b in that interval. By the mean value theorem applied to $[a,b]$, there would be c in (a,b) such that $f'(c) = \frac{f(b) - f(a)}{b - a} \neq 0$.

Armin Straub
straub@southalabama.edu

If $f'(x) = g'(x)$ for all x in some interval, then f and g differ by a constant on that interval.

Example 102. Suppose $f'(x) = 4x$ for all x.

(a) What can we say about $f(x)$?

(b) What is $f(x)$ is $f(1) = 5$?

Solution.

(a) $f(x) = 2x^2 + C$

(b) Using $f(x) = 2x^2 + C$, we get $f(1) = 2 + C$ so that solving $2 + C = 5$ results in $C = 3$. Hence, $f(x) = 2x^2 + 3$.

Example 103. Find all possible $f(x)$ such that $f'(x) = x^3 + 2$.

Solution. $f(x) = \frac{1}{4}x^4 + 2x + C$

Comment. We are computing the **antiderivative** of $x^3 + 2$ in this problem. We'll return to this notion!