Example 81.

- The area of a circle of radius r is $A = \pi r^2$.
  The circumference of that circle is $2\pi r$.
- The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.
  The surface area of that sphere is $4\pi r^2$.

Note that the circumference is $\frac{\mathrm{d}A}{\mathrm{d}r}$ and that the surface area is $\frac{\mathrm{d}V}{\mathrm{d}r}$. Can you say why?

See: https://www.math.hmc.edu/funfacts/ffiles/20004.2-3.shtml

Linearization

The **linearization** of $f(x)$ at $x = a$ is the tangent line

$$L(x) = f(a) + f'(a)(x - a).$$

It serves as an approximation ("standard linear approximation") of f near a.

Example 82. Find the linearization of $f(x) = x^3(2x+1)^{10}$ at $x = -1$.

Solution. $f'(x) = 3x^2(2x+1)^{10} + x^3 \cdot 10(2x+1)^9 \cdot 2 = 3x^2(2x+1)^{10} + 20x^3(2x+1)^9$ and $f'(-1) = 23$.
Since $f(-1) = -1$, the linearization therefore is $L(x) = -1 + 23(x+1)$.

Example 83. Find a linearization of $f(x) = \ln(x)$ at a suitable integer x near 1.01 and use that to estimate $f(1.01)$ without a calculator.

Solution. The only suitable point near 1.01 is 1. Using $f'(x) = \frac{1}{x}$, we have $f(1) = \ln(1) = 0$ and $f'(1) = 1$.
Hence $L(x) = 0 + 1(x-1) = x - 1$ is the linear approximation of $f(x)$ at $x = 1$.
Using that, we estimate $f(1.01) \approx L(1.01) = 0.01$.
For comparison. The exact value is $f(1.01) = \ln(1.01) = 0.00995033\ldots$.
Make a rough sketch. Since the tangent line at $x = 1$ lies above the graph of $f(x)$, we know that our approximation is an overestimate.
Comment. Techniques like this are indeed used by calculators to evaluate transcendental functions.

Example 84. Approximate $e^{-0.0015}$ using an appropriate linear approximation.

Solution. We use the linearization of $f(x) = e^x$ at $x = 0$. Since $f(0) = 1$ and $f'(0) = 1$, the linearization is $L(x) = 1 + 1(x - 0) = x + 1$.
Using that, we estimate $e^{-0.0015} = f(-0.0015) \approx L(-0.0015) = 1 - 0.0015 = 0.9985$.
For comparison. From a sketch, we know that our estimate is an underestimate (because the tangent line is below the graph). The exact value is $e^{-0.0015} = 0.998501\ldots$.

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