Quiz #6 (Tuesday)

- implicit differentiation (tangent and normal line to a curve)
- compute derivatives involving log's and inverse trig functions

Review. Implicit differentiation and derivatives of inverse functions

Related rates

Example 78. Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Determine an equation that relates $\frac{\mathrm{d}r}{\mathrm{d}t}$ to $\frac{\mathrm{d}A}{\mathrm{d}t}$.

Solution. Applying $\frac{\mathrm{d}}{\mathrm{d}t}$ to both sides of $A = \pi r^2$, we find $\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \frac{\mathrm{d}r}{\mathrm{d}t}$.

Comment. Make a sketch of two circles, one with larger radius than the other. For which of the two is the area increasing more when the radius r is increased by $\mathrm{d}r$ (so that the new radius is $r + \mathrm{d}r$)? See how this is reflected by the formula relating $\frac{\mathrm{d}A}{\mathrm{d}t}$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$.

Example 79. Consider a cone with base a circle of radius r and with height h. Recall that the volume of this cone is $V = \frac{1}{3}\pi r^2 h$. (Suppose r and h are differentiable functions of t.)

(a) How is $\frac{\mathrm{d}V}{\mathrm{d}t}$ related to $\frac{\mathrm{d}h}{\mathrm{d}t}$ if r is constant?

(b) How is $\frac{\mathrm{d}V}{\mathrm{d}t}$ related to $\frac{\mathrm{d}r}{\mathrm{d}t}$ if h is constant?

(c) How is $\frac{\mathrm{d}V}{\mathrm{d}t}$ related to $\frac{\mathrm{d}h}{\mathrm{d}t}$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$ if neither h nor r are constant?

Solution. In each case, we apply $\frac{\mathrm{d}}{\mathrm{d}t}$ to both sides of $V = \frac{1}{3}\pi r^2 h$.

(a) If r is constant, then we get $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi r^2 \frac{\mathrm{d}h}{\mathrm{d}t}$.

(b) If h is constant, then we get $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{2}{3}\pi r h \frac{\mathrm{d}r}{\mathrm{d}t}$.

(c) In general, we get $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi r^2 \frac{\mathrm{d}h}{\mathrm{d}t} + \frac{2}{3}\pi r h \frac{\mathrm{d}r}{\mathrm{d}t}$. (This is nothing but the product rule.)

Note. If, for instance, h is constant, then $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$ and we obtain the earlier formula.

Example 80. A baseball diamond is a square with side $90\,\mathrm{ft}$. A batter hits the ball and runs towards first base with a speed of $24\,\mathrm{ft/sec}$.

At what rate is his distance from second base decreasing when he is halfway to first base?

Solution. Let us introduce coordinates: home plate $H = (0,0)$, 1st base $B_1 = (90,0)$, 2nd base $B_2 = (90,90)$.
Let x be the distance of the batter from H. In other words, his position is $(x,0)$.
Let r be the distance from the batter to B_2. Then $r^2 = (90-x)^2 + 90^2$.
Observe that we want $\frac{\mathrm{d}r}{\mathrm{d}t}$ and that we know $\frac{\mathrm{d}x}{\mathrm{d}t} = 24$.
We apply $\frac{\mathrm{d}}{\mathrm{d}t}$ to both sides of $r^2 = (90-x)^2 + 90^2$ to get $2r\frac{\mathrm{d}r}{\mathrm{d}t} = -2(90-x)\frac{\mathrm{d}x}{\mathrm{d}t}$. So, $\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{90-x}{r}\frac{\mathrm{d}x}{\mathrm{d}t}$.
When the batter is halfway to B_1, we have $x = 45$ and $r = \sqrt{(90-x)^2 + 90^2} = 45\sqrt{5}$.
Hence, $\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{90-x}{r}\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{90-45}{45\sqrt{5}} \cdot 24 = -\frac{24}{\sqrt{5}} \approx -10.73\,\mathrm{ft/sec}$.
In other words, the batter's distance to B_2 is decreasing by $10.73\,\mathrm{ft/sec}$.

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