- $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x)$ **(chain rule)**

In Leibniz notation: if $y = f(u)$ and $u = g(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$

Why? $\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \overset{h\to 0}{\longrightarrow} f'(g(x)) \cdot g'(x)$

[to make this into a proper proof, we need to worry about the possibility that $g(x+h) - g(x) = 0$ for certain h; if g is locally increasing or decreasing at x, then this is no concern and the above proves the chain rule]

Example 62. $\frac{\mathrm{d}}{\mathrm{d}x} e^{-x}$

Solution. (without chain rule) $\frac{\mathrm{d}}{\mathrm{d}x} e^{-x} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{e^x} = \frac{0 \cdot e^x - 1 \cdot e^x}{(e^x)^2} = -\frac{e^x}{e^{2x}} = -e^{-x}$

Solution. (with chain rule) Write $h(x) = e^{-x}$ as $h(x) = f(g(x))$ with $f(u) = e^u$ and $g(x) = -x$.
$\frac{\mathrm{d}}{\mathrm{d}x} e^{-x} = h'(x) = f'(g(x))g'(x) = e^{g(x)} \cdot (-1) = -e^{-x}$

Example 63.

(a) $\frac{\mathrm{d}}{\mathrm{d}x} \cos^5(x)$ (b) $\frac{\mathrm{d}}{\mathrm{d}x} \cos^5(7x)$ (c) $\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{2 + \cos^5(7x)}$

Solution.

(a) Write $h(x) = \cos^5(x)$ as $h(x) = f(g(x))$ with $f(u) = u^5$ and $g(x) = \cos(x)$.
$\frac{\mathrm{d}}{\mathrm{d}x} \cos^5(x) = f'(g(x))g'(x) = 5g(x)^4 g'(x) = 5\cos^4(x) \cdot (-\sin(x)) = -5\cos^4(x)\sin(x)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} \cos^5(7x) = 5\cos^4(7x) \frac{\mathrm{d}}{\mathrm{d}x}\cos(7x) = = 5\cos^4(7x)\left(-\sin(7x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}[7x]\right) = -35\cos^4(7x)\sin(7x)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{2 + \cos^5(7x)} = \frac{1}{2\sqrt{2 + \cos^5(7x)}} \frac{\mathrm{d}}{\mathrm{d}x}[2 + \cos^5(7x)] = \frac{-35\cos^4(7x)\sin(7x)}{2\sqrt{2 + \cos^5(7x)}}$
[in the last step, we used our result from the previous item]

Example 64. (derivatives of exponentials) Note that $a^x = e^{\ln(a^x)} = e^{x\ln(a)}$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x\ln(a)} = e^{x\ln(a)} \frac{\mathrm{d}}{\mathrm{d}x}[x\ln(a)] = \ln(a)\, a^x.$$

Example 65. The position of a particle moving along a line is $s = \sqrt{1 + 4t}$, where s is in meters and t in seconds. Find the particle's velocity and acceleration at $t = 2$.

Solution. By definition, velocity is $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ (in $\frac{\mathrm{m}}{\mathrm{sec}}$) and acceleration is $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ (in $\frac{\mathrm{m}}{\mathrm{sec}^2}$).

Using the chain rule, $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{4}{2\sqrt{1 + 4t}} = \frac{2}{\sqrt{1 + 4t}}$ and $\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{2}{\sqrt{1 + 4t}} = \frac{2 \cdot \left(-\frac{1}{2}\right) \cdot 4}{(1 + 4t)^{3/2}} = \frac{-4}{(1 + 4t)^{3/2}}$.

The velocity at $t = 2$ is $\left[\frac{\mathrm{d}s}{\mathrm{d}t}\right]_{t=2} = \left[\frac{2}{\sqrt{1 + 4t}}\right]_{t=2} = \frac{2}{\sqrt{9}} = \frac{2}{3}\frac{\mathrm{m}}{\mathrm{sec}}$.

The acceleration at $t = 2$ is $\left[\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\right]_{t=2} = \left[\frac{-4}{(1 + 4t)^{3/2}}\right]_{t=2} = \frac{-4}{9^{3/2}} = -\frac{4}{27}\frac{\mathrm{m}}{\mathrm{sec}}$.

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