Derivatives

Review.

- The slope of a line through (x_0, y_0) and (x_1, y_1) is $m = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0}$.
- The line through (x_0, y_0) with slope m has the equation $y - y_0 = m(x - x_0)$.
  [Note how this equation is just $m = \frac{x - x_0}{y - y_0}$.]

Definition 46. The **derivative** of $y = f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Important comments.

- If the limit defining $f'(x_0)$ exists, then we say that $f(x)$ is **differentiable** at $x = x_0$.
- Other common notations include: $f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \dot{y} = f_x = Df(x) = D_x f(x)$
- $\frac{f(x+h) - f(x)}{h}$ is the slope through $(x, f(x))$ and $(x+h, f(x+h))$, two points on the curve $y = f(x)$. The corresponding line is called a **secant line**. As $h \to 0$, the two points merge into one.

The value $f'(x_0)$ is

- the slope of (the line tangent to) the curve $y = f(x)$ at $x = x_0$,
- the rate of change of $f(x)$ at $x = x_0$.

Example 47. If $f(x)$ describes the distance in mi travelled by an object after time x in h. What is measured by $f'(x)$ and what are the units?

Solution. The units of $f'(x)$ are $\frac{\mathrm{mi}}{\mathrm{h}}$. (Why?!) This is the velocity of the object.

Example 48. (as in midterm!)

(a) Compute $f'(x)$ for $f(x) = x^2 + 1$.

(b) Determine the line tangent to the graph of $f(x)$ at $x = 3$.

Solution.

(a) We need to determine $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for $f(x) = x^2 + 1$.
Since $f(x+h) = (x+h)^2 + 1 = x^2 + 2hx + h^2 + 1$, we have

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x^2 + 2hx + h^2 + 1) - (x^2 + 1)}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

(b) From the first part, the slope of that line is $f'(3) = 6$. It also passes through $(3, f(3)) = (3, 10)$. Hence, it has the equation $y - 10 = 6(x - 3)$, which simplifies to $y = 6x - 8$.