(ε-δ definition of limits) $\lim_{x \to c} f(x) = L$

$\Longleftrightarrow$ For every $\varepsilon > 0$ there is $\delta > 0$ such that, for all x, $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.

Make a sketch as we did in class! Test yourself by explaining the (rather abstract!) definition using the sketch. You can find lots of nice sketches and explanations in Section 2.3 of our book.

Example 43. Make a sketch of $f(x) = \sqrt{x}$. We know that $\lim_{x \to 4} f(x) = 2$.

From that sketch, determine the largest value of δ such that $|x - 4| < \delta$ implies $|f(x) - 2| < 0.1$.

Solution. $|f(x) - 2| < 0.1$ means $1.9 < f(x) < 2.1$. Note that $1.9^2 = 3.61$ and that $f(3.61) = 1.9$. Likewise, $2.1^2 = 4.41$ and $f(4.41) = 2.1$. Mark the points $(3.61, 1.9)$ and $(4.41, 2.1)$ in your plot, as well as the corresponding intervals on the x and y-axis.

Note that $3.61 < x < 4.41$ (equivalently, $4 - 0.39 < x < 4 + 0.41$) guarantees that $1.9 < f(x) < 2.1$. Hence, the largest δ is $\delta = 0.39$.

Likewise. Repeat all this with $\varepsilon = 0.01$ (instead of $\varepsilon = 0.1$). In that case, $3.9601 = 4 - 0.0399 < x < 4.0401$ guarantees that $1.99 < f(x) < 2.01$. It follows that the largest δ now is $\delta = 0.0399$.

Example 44. Let $f(x) = \frac{1}{x}$. We know that $\lim_{x \to 2} f(x) = \frac{1}{2}$.

Given $\varepsilon = \frac{1}{100}$, determine the largest value of δ such that $|x - 2| < \delta$ implies $\left| f(x) - \frac{1}{2} \right| < \varepsilon$.

Solution. $\left| \frac{1}{x} - \frac{1}{2} \right| < \frac{1}{100}$ is equivalent to

$\Longleftrightarrow$ $-\frac{1}{100} < \frac{1}{x} - \frac{1}{2} < \frac{1}{100}$

$\Longleftrightarrow$ $\frac{49}{100} < \frac{1}{x} < \frac{51}{100}$

$\Longleftrightarrow$ $\frac{100}{49} > x > \frac{100}{51}$

$\Longleftrightarrow$ $\frac{100}{51} < x < \frac{100}{49}$ or, rounded to three digits, $1.961 < x < 2.041$.

In other words, $2 - 0.039 < x < 2 + 0.041$.

This means that the largest δ is $\delta = 0.039$.

Alternative (but ultimately equivalent) approach. Note that $f(x)$ is decreasing around $x = 2$. Since we want $\left| f(x) - \frac{1}{2} \right| < \varepsilon$, we can solve the equations $f(x) = \frac{1}{2} + \varepsilon$ (solved by $x = \frac{100}{51}$) and $f(x) = \frac{1}{2} - \varepsilon$ (solved by $x = \frac{100}{49}$) to find for which x we are right on the boundary of that inequality. The conclusion would be that the inequality holds for $\frac{100}{51} < x < \frac{100}{49}$, leading to the same δ. Realize that this actually is exactly the argument from our earlier solution (except that we were a bit sloppy about the inequalities for x, which we justified using the fact that $f(x)$ is decreasing).

(ε-δ definition of continuity) $f(x)$ is continuous at $x = c$ (that is, $\lim_{x \to c} f(x) = f(x)$)

$\Longleftrightarrow$ For every $\varepsilon > 0$ there is $\delta > 0$ so that, for all x, $0 < |x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

This is roughly saying that a function is continuous if small variations in the output result can be guaranteed through small variations in the input.

Example 45. (exam prep) Determine the following limits:

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right), \quad \lim_{x \to \infty} \sin\left(\frac{1}{x}\right), \quad \lim_{x \to 0} x \sin\left(\frac{1}{x}\right), \quad \lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$$

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