Quiz #3 (Tuesday)

- which parameter values make a function continuous
- intermediate value theorem

Example 33. Let $f(x)$ be a complicated continuous function taking the following values:

x	-2	-1	0	1	2
$f(x)$	4	5	-1	-3	4

Using the intermediate value theorem, what can we conclude about solutions to the equation $f(x)=0$?

Solution. We know that there is a solution x in the interval $[-1,0]$ and another in $[1,2]$ (and there could certainly be more solutions).

Important note. Note that we cannot say anything about whether there is a solution in $[-2,-1]$. (It would be incorrect to say that there is no solution in $[-2,-1]$.)

Limits involving infinity and asymptotes

Good news. All limit laws for $\lim_{x\to c} f(x)$ continue to hold if $c=\infty$ or $c=-\infty$.

Example 34. $\lim_{x\to\infty}\frac{1}{x}=0$ and $\lim_{x\to-\infty}\frac{1}{x}=0$.

Make a sketch!

Example 35. Determine $\lim_{x\to 0}\frac{1}{x}$, $\lim_{x\to 0^-}\frac{1}{x}$ and $\lim_{x\to 0^+}\frac{1}{x}$.

Solution. $\lim_{x\to 0^-}\frac{1}{x}=-\infty$ and $\lim_{x\to 0^+}\frac{1}{x}=\infty$. Since the two differ, $\lim_{x\to 0}\frac{1}{x}$ does not exist.

Example 36. Determine the following limits:

(a) $\lim_{x\to\infty}\frac{4x^3-7}{5x^2+6x+3}$

(b) $\lim_{x\to\infty}\frac{4x^3-7}{5x^3+6x+3}$

(c) $\lim_{x\to\infty}\frac{4x^3-7}{5x^4+6x+3}$

Important comment. Note how, in each case, the limits are of the indeterminate form "$\frac{\infty}{\infty}$" (that is, both numerator and denominator approach ∞ as $x\to\infty$). As this example illustrates, this tells us nothing.

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Solution. (using limit laws) In each case, we divide both numerator and denominator by the largest power of x that appears in the denominator.

(a) Dividing by x^2, we see that $\frac{4x^3-7}{5x^2+6x+3} = \frac{4x - \frac{7}{x^2}}{5 + \frac{6}{x} + \frac{3}{x^2}} \overset{\text{as } x\to\infty}{\longrightarrow} \frac{\infty - 0}{5+0+0} = \infty$.

In other words, $\lim_{x\to\infty} \frac{4x^3-7}{5x^2+6x+3} = \infty$.

[Note how, in the final step, we are using the limit laws for sums and quotients.]

(b) Dividing by x^3, we see that $\frac{4x^3-7}{5x^3+6x+3} = \frac{4 - \frac{7}{x^3}}{5 + \frac{6}{x^2} + \frac{3}{x^3}} \overset{\text{as } x\to\infty}{\longrightarrow} \frac{4-0}{5+0+0} = \frac{4}{5}$.

Comment. We say that $f(x) = \frac{4x^3-7}{5x^3+6x+3}$ has the line $y = \frac{4}{5}$ as a **horizontal asymptote**.

(c) Dividing by x^4, we see that $\frac{4x^3-7}{5x^4+6x+3} = \frac{\frac{4}{x} - \frac{7}{x^4}}{5 + \frac{6}{x^3} + \frac{3}{x^4}} \overset{\text{as } x\to\infty}{\longrightarrow} \frac{0-0}{5+0+0} = 0$.

Comment. Thus $f(x) = \frac{4x^3-7}{5x^4+6x+3}$ has the line $y = 0$ as a horizontal asymptote.

Solution. (advanced) Once we understand what we are doing, we realize that the approach in the first solution is leading to the conclusion that we can focus on the leading terms only:

(a) $\lim_{x\to\infty} \frac{4x^3-7}{5x^2+6x+3} = \lim_{x\to\infty} \frac{4x^3}{5x^2} = \lim_{x\to\infty} \frac{4}{5}x = \infty$

(b) $\lim_{x\to\infty} \frac{4x^3-7}{5x^3+6x+3} = \lim_{x\to\infty} \frac{4x^3}{5x^3} = \lim_{x\to\infty} \frac{4}{5} = \frac{4}{5}$

(c) $\lim_{x\to\infty} \frac{4x^3-7}{5x^4+6x+3} = \lim_{x\to\infty} \frac{4x^3}{5x^4} = \lim_{x\to\infty} \frac{4}{5x} = 0$

Theorem 37. (sandwich theorem) Suppose $g(x) \leqslant f(x) \leqslant h(x)$ for x close enough to c.
If $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$, then $\lim_{x\to c} f(x) = L$.

Why? Both $g(x)$ and $h(x)$ approach L as $x \to c$. Since $f(x)$ is "sandwiched" between the two, it has no choice but to approach L as well.

Example 38. Determine $\lim_{x\to\infty} \frac{\sin(2x)}{x}$.

Solution. Note that $-\frac{1}{x} \leqslant \frac{\sin(2x)}{x} \leqslant \frac{1}{x}$ (for $x > 0$).

Since $\lim_{x\to\infty} -\frac{1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x}$, the sandwich theorem implies that $\lim_{x\to\infty} \frac{\sin(2x)}{x} = 0$.

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