Review. $f(x) = \frac{\sin(x)}{x}$ is continuous for all $x \neq 0$. Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1$, defining $f(0) = 1$ results in an extension, which is continuous for all x.

Example 30. For what values of a is $f(x) = \begin{cases} 2x - 1, & x < 3, \\ ax^2 + 1, & x \geqslant 3, \end{cases}$ a continuous function?

Solution. Observe that $f(x)$ is always continuous at every point except, possibly, $x = 3$. (Why?!)
In order for $f(x)$ to be continuous at $x = 3$, we need $\lim_{x \to 3} f(x) = f(3)$.
We have $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2x - 1) = 2 \cdot 3 - 1 = 5$ whereas $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (ax^2 + 1) = 9a + 1 = f(3)$.
Hence, $\lim_{x \to 3} f(x) = f(3)$ if and only if $5 = 9a + 1$, which happens if and only if $a = \frac{4}{9}$.
Thus, $f(x)$ is continuous if and only if $a = \frac{4}{9}$.

Intermediate value theorem

Theorem 31. (intermediate value theorem) If f is continuous on $[a, b]$ and y_0 is between $f(a)$ and $f(b)$, then $f(c) = y_0$ for some c in $[a, b]$.

Why? Make a sketch! And keep in mind that the graph of a continuous function has no gaps or jumps.

Example 32. Show that $x^3 + 3x^2 - 1 = 0$ has two solutions in the interval $[-1, 1]$.

Solution. Let $f(x) = x^3 + 3x^2 - 1$. Then $f(-1) = 1$ and $f(1) = 3$. On the other hand, $f(0) = -1$.
By the intermediate value theorem, it follows that there is c in $[-1, 0]$ such that $f(c) = 0$ (because 0 is between $f(-1) = 1$ and $f(0) = -1$). Likewise, there is c in $[0, 1]$ such that $f(c) = 0$ (because 0 is between $f(0) = -1$ and $f(1) = 3$).

Extra. Similarly, we could show that $x^3 + 3x^2 - 3 = 0$ has three solutions in the interval $[-3, 1]$. Indeed, note that $f(-3) = -1$. Hence, there is c in $[-3, -1]$ such that $f(c) = 0$ (because 0 is between $f(-3) = -1$ and $f(-1) = 1$).

Comment. Recall that a cubic has at most three roots, so we must have found all of them.

Comment. Note that we can find appropriate values (here, we used $x = -3, -1, 0, 1$) by inspecting a graph (even if it isn't particularly accurate). Finding the exact solutions is much more involved (just for fun, the solution in $[0, 1]$ is $2\cos(2\pi/9) - 1 \approx 0.532$).

Advanced comment. Over the complex numbers, and taking multiplicity into account, a degree n equation has exactly n solutions.