Example 5. GAP has 40% off and there is a coupon for an additional 20% off. What is the total savings?

Solution. $0.6 \cdot 0.8 = 0.48$, so that the savings are 52%.

Example 6. Determine the domain and range of $f(x) = \frac{5}{1 - e^{3x}}$.

Review. $f(x)$ is a composition of two simpler functions! Namely, $f(x) = g(h(x))$ where $g(z) = \frac{5}{1 - z}$ and $h(x) = e^{3x}$. Recall that we also write $f = g \circ h$.

Note. Other good choices are possible: for instance, $g(z) = \frac{5}{z}$ and $h(x) = 1 - e^{3x}$.

Solution. The domain of $f(x)$ consists of all x such that $1 - e^{3x} \neq 0$. Since $1 - e^{3x} = 0$ has only the solution $x = 0$, the domain of $f(x)$ is $\{x : x \neq 0\}$ or, in interval notation, $(-\infty, 0) \cup (0, \infty)$.

Since e^{3x} has range $(0, \infty)$, the range of $f(x)$ is the same as the range of $\frac{5}{1 - z}$ with z restricted to $(0, \infty)$.

Make a plot of $\frac{5}{1 - z}$! (How can it be obtained from the familiar plot of $\frac{1}{z}$?)

From that plot, we conclude that the range of $f(x)$ is $(-\infty, 0) \cup (5, \infty)$.

Example 7. $\cos(x)$, $\sin(x)$ and $\tan(x)$ are not 1-1. What do we do to define their inverses $\arccos(x)$, $\arcsin(x)$, $\arctan(x)$ nevertheless?

Solution. We restrict the domain of these functions, so that they become 1-1.

Specifically, we restrict $\cos(x)$ to $x \in [0, \pi]$. And $\sin(x)$ to $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. And $\tan(x)$ to $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, as well.

Make sketches!

Example 8. Suppose we have capital 1 and that, annually, we are receiving $1 = 100\%$ interest. How much capital do we have at the end of a year, if $\frac{1}{n}$ interest is paid n times a year?

[For instance, $n = 12$ if we receive monthly interest payments.]

Solution. At the end of the year, we have $\left(1 + \frac{1}{n}\right)^n$.

For instance. If $n = 1$, we will have $(1 + 1)^1 = 2$.

If $n = 2$, we will have $\left(1 + \frac{1}{2}\right)^2 = \frac{9}{4} = 2.25$.

If $n = 12$, we will have $\left(1 + \frac{1}{12}\right)^{12} \approx 2.613$.

If $n = 365$, we will have $\left(1 + \frac{1}{365}\right)^{365} \approx 2.715$.

If we keep increasing n, then we will get closer and closer to $e = 2.718281828459045\ldots$

We will learn how to compute such **limits**. The limit here is: $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$.

Comment. Though, obviously, the choice of n makes a big difference, in each case the **APR** (annual percentage rate) is $\frac{1}{n} \cdot n = 1$. On the other hand, the **APY** (annual percentage yield) takes compounding interest into account.

Example 9. (extra) $\log_2 6 - \log_2 15 + \log_2 20 = \log_2 \frac{6 \cdot 20}{15} = \log_2 8 = 3$

Example 10. (extra) $\ln(a + b) + \ln(a - b) - 2\ln(c) = \ln \frac{a^2 - b^2}{c^2}$

Example 11. (extra) Find the inverse of $f(x) = x^2 - x$, with the domain restricted to $x \geqslant 1/2$.

Solution. We solve $y = x^2 - x$ for x, and find $x = \frac{1 \pm \sqrt{1 + 4y}}{2}$.

Since $x \geqslant 1/2$, we take "$+$" and conclude that $f^{-1}(x) = \frac{1 + \sqrt{1 + 4x}}{2}$.

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