Practice for Final Exam

MATH 125 — Calculus 1
Monday, April 29

Please print your name:

Besides the allowed calculator, no notes or tools of any kind will be permitted.

The final exam is cumulative. The problems below only cover the material since Midterm #3.

- Start by doing the practice problems for Midterm #1, #2 and #3, as well as the problems below.
- Then, retake all quizzes. (Versions with and without solutions are posted to our course website.)
- Finally, retake Midterm #1, #2 and #3.

Problem 1. Compute the following derivatives.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{5}\sin(t^2+1)\mathrm{d}t$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sqrt{x}}^{4\sqrt{x}}\sin(t^2+1)\mathrm{d}t$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{3x^2}^{5x^2}\cos(\sin(t))\mathrm{d}t$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{1/x}^{x}\cos(\sin(t))\mathrm{d}t$

Solution. Note that $\int_a^b f(t)\mathrm{d}t = \int_m^b f(t)\mathrm{d}t - \int_m^a f(t)\mathrm{d}t$ for any m (provided that a, b, m are in an interval I on which f is continuous).

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{5}\sin(t^2+1)\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}x}\int_{5}^{x^2}\sin(t^2+1)\mathrm{d}t = -2x\sin(x^4+1)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sqrt{x}}^{4\sqrt{x}}\sin(t^2+1)\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^{4\sqrt{x}}\sin(t^2+1)\mathrm{d}t - \int_a^{\sqrt{x}}\sin(t^2+1)\mathrm{d}t\right] = 4\cdot\frac{1}{2\sqrt{x}}\sin((4\sqrt{x})^2+1) - \frac{1}{2\sqrt{x}}\sin(x+1)$
$= \frac{2}{\sqrt{x}}\sin(16x+1) - \frac{1}{2\sqrt{x}}\sin(x+1)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{3x^2}^{5x^2}\cos(\sin(t))\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^{5x^2}\cos(\sin(t))\mathrm{d}t - \int_a^{3x^2}\cos(\sin(t))\mathrm{d}t\right] = 10x\cos(\sin(5x^2)) - 6x\cos(\sin(3x^2))$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{1/x}^{x}\cos(\sin(t))\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^{x}\cos(\sin(t))\mathrm{d}t - \int_a^{1/x}\cos(\sin(t))\mathrm{d}t\right] = \cos(\sin(x)) - \cos\left(\sin\left(\frac{1}{x}\right)\right)\cdot\left(-\frac{1}{x^2}\right)$
$= \cos(\sin(x)) + \frac{1}{x^2}\cos\left(\sin\left(\frac{1}{x}\right)\right)$ □

Problem 2.

(a) Find the net area between the x-axis and $f(x) = x^3 - 4x$ for x in $[-1, 3]$.

(b) Find the total area between the x-axis and $f(x) = x^3 - 4x$ for x in $[-1, 3]$.

Solution.

(a) The net area is: $\int_{-1}^{3}(x^3-4x)\mathrm{d}x = \left[\frac{1}{4}x^4 - 2x^2\right]_{-1}^{3} = \frac{9}{4} - \left(-\frac{7}{4}\right) = 4$

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(b) Note that $f(x)=x(x^2-4)=x(x-2)(x+2)$, so that $f(x)=0$ for $x=-2,0,2$.

Hence, we conclude that $f(x)\leqslant 0$ for $x\leqslant -2$, $f(x)\geqslant 0$ for x in $[-2,0]$, $f(x)\leqslant 0$ for x in $[0,2]$ and $f(x)\geqslant 0$ for $x\geqslant 2$.

Consequently, the total area is: $\int_{-1}^{0} f(x)\mathrm{d}x - \int_{0}^{2} f(x)\mathrm{d}x + \int_{2}^{3} f(x)\mathrm{d}x$

$$=\left[\frac{1}{4}x^4-2x^2\right]_{-1}^{0}-\left[\frac{1}{4}x^4-2x^2\right]_{0}^{2}+\left[\frac{1}{4}x^4-2x^2\right]_{2}^{3}=\left(0-\left(-\frac{7}{4}\right)\right)-(-4-0)+\left(\frac{9}{4}-(-4)\right)=12$$

Comment. Equivalently, the total area is $\int_{-1}^{3}|x^3-4x|\mathrm{d}x$. □

Problem 3. Let $f(x)=x^3-x^2-2x$.

(a) What are the minimum, maximum and average value of $f(x)$ for x in $[-1,3]$?

(b) What are the minimum, maximum and average value of $f(x)$ for x in $[-1,1]$?

Solution. Since $f'(x)=3x^2-2x-2$ has roots $\frac{1}{3}(1\pm\sqrt{7})$, the critical points of $f(x)$ are $\frac{1}{3}(1-\sqrt{7})\approx -0.549$ and $\frac{1}{3}(1+\sqrt{7})\approx 1.215$.

(a) The average value is:

$$\frac{1}{3-(-1)}\int_{-1}^{3} f(x)\mathrm{d}x=\frac{1}{4}\int_{-1}^{3}(x^3-x^2-2x)\mathrm{d}x=\frac{1}{4}\left[\frac{1}{4}x^4-\frac{1}{3}x^3-x^2\right]_{-1}^{3}=\frac{1}{4}\left(\frac{9}{4}-\left(-\frac{5}{12}\right)\right)=\frac{2}{3}$$

Minimum and maximum have to occur either at an endpoint ($x=-1$ or $x=3$) or at a critical point ($x=\frac{1}{3}(1-\sqrt{7})$ or $x=\frac{1}{3}(1+\sqrt{7})$). Since $f(-1)=0$, $f(3)=12$, $f\left(\frac{1}{3}(1-\sqrt{7})\right)\approx 0.631$, $f\left(\frac{1}{3}(1+\sqrt{7})\right)\approx -2.113$, the minimum is $f\left(\frac{1}{3}(1+\sqrt{7})\right)\approx -2.113$ and the maximum is $f(3)=12$.

(b) The average value is:

$$\frac{1}{1-(-1)}\int_{-1}^{1} f(x)\mathrm{d}x=\frac{1}{2}\int_{-1}^{1}(x^3-x^2-2x)\mathrm{d}x=\frac{1}{2}\left[\frac{1}{4}x^4-\frac{1}{3}x^3-x^2\right]_{-1}^{1}=\frac{1}{2}\left(-\frac{13}{12}-\left(-\frac{5}{12}\right)\right)=-\frac{1}{3}$$

Minimum and maximum have to occur either at an endpoint ($x=-1$ or $x=1$) or at a critical point (only $x=\frac{1}{3}(1-\sqrt{7})$ is in the interval $[-1,1]$). Since $f(-1)=0$, $f(1)=-2$, $f\left(\frac{1}{3}(1-\sqrt{7})\right)\approx 0.631$, the minimum is $f(1)=-2$ and the maximum is $f\left(\frac{1}{3}(1-\sqrt{7})\right)\approx 0.631$. □