# Gessel-Lucas congruences for sporadic sequences

## Special Session on Modular Forms in Combinatorics and Number Theory
## AMS Fall Southeastern Sectional Meeting, Tulane University


**Armin Straub**

October 5, 2025

University of South Alabama

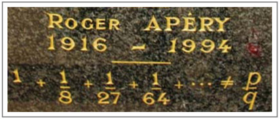

**THM** Lucas 1878

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots \pmod{p}$$

where $n_i$ and $k_i$ are the base $p$ digits of $n$ and $k$.

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$= \operatorname{diag} \frac{1}{(1-x-y)(1-z-w)-xyzw}$$




Slides available at:
`http://arminstraub.com/talks`

# Lucas congruences

**THM** Lucas 1878

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots \pmod{p},$$

where $n_i$ and $k_i$ are the $p$-adic digits of $n$ and $k$.

**EG**

$$\binom{136}{79} \equiv \binom{3}{2}\binom{5}{4}\binom{2}{1} = 3 \cdot 5 \cdot 2 \equiv 2 \pmod{7}$$

LHS $= 1009220746942993946271525627285911932800$

# Lucas congruences

**THM** Lucas 1878

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots \pmod{p},$$

where $n_i$ and $k_i$ are the $p$-adic digits of $n$ and $k$.

**EG**

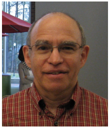
$$\binom{136}{79} \equiv \binom{3}{2}\binom{5}{4}\binom{2}{1} = 3\cdot 5\cdot 2 \equiv 2 \pmod{7}$$

LHS $= 1009220746942993946271525627285911932800$

- Interesting sequences like the **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy such **Lucas congruences** as well:



**THM** Gessel '82

$$A(n) \equiv A(n_0)A(n_1)\cdots A(n_r) \pmod{p}$$

- Equivalently: $A(pn+k) \equiv A(n)A(k) \pmod{p}$

Here and elsewhere: $0 \leqslant k < p$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2 + 17n + 5) u_n - n^3 u_{n-1}.$$

**THM** Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

**THM** Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

**proof** The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

# Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

**THM** Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

**Q** Beukers, Zagier, Almkvist, Zudilin, Cooper: Are there other tuples $(a, b, c)$ for which the recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - cn^3 u_{n-1}.$$

has an integral solution?

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)
- $6+6+3$ **sporadic sequences** known.

# The six (basic) sporadic Apéry-like numbers of order $3$

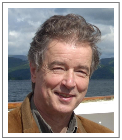
$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - cn^3 u_{n-1}$$

| $(a,b,c)$ | $A(n)$ | |
|---|---|---|
| $(17,5,1)$ | $\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$ | Apéry numbers |
| $(12,4,16)$ | $\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$ | Kauers–Zeilberger diagonal |
| $(10,4,64)$ | $\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$ | Domb numbers |
| $(7,3,81)$ | $\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$ | Almkvist–Zudilin numbers |
| $(11,5,125)$ | $\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$ | |
| $(9,3,-27)$ | $\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$ | |

# Modularity of Apéry-like numbers

- Beukers ('87) observed that the Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$



satisfy:

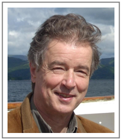
$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

# Modularity of Apéry-like numbers

- Beukers ('87) observed that the Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$



satisfy:

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

**FACT** Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

| | |
|---|---|
| $f(\tau)$ | modular form of weight $k$ |
| $x(\tau)$ | modular function |
| $y(x)$ | such that $y(x(\tau)) = f(\tau)$ |

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

# Gessel–Lucas congruences

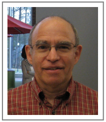
- Lucas congruences: $A(pn+k) \equiv A(n)A(k) \pmod{p}$

**THM** Malik–S '16: All of the $6+6+3$ known sporadic sequences satisfy Lucas congruences modulo every prime. (Proof long and technical for 2 sequences)

# Gessel–Lucas congruences

- Lucas congruences: $A(pn+k) \equiv A(n)A(k) \pmod{p}$

**THM** Malik–S '16 — All of the $6+6+3$ known sporadic sequences satisfy Lucas congruences modulo every prime. (Proof long and technical for 2 sequences)



- In the case of the Apéry numbers, Gessel ('82) observed that these congruences can be extended modulo $p^2$.

**THM** S '24 — All of the $6+6+3$ known sporadic sequences satisfy **Gessel–Lucas congruences** modulo every odd prime:

$$A(pn+k) \equiv A(k)A(n) + pnA'(k)A(n) \pmod{p^2}$$

- Here, $A'(n)$ is the formal derivative of $A(n)$.
  These are rational numbers!

# The formal derivative of recurrence sequences

- Suppose $A(n)$ is the unique solution for all $n \geqslant 0$ to

$$\sum_{j=0}^{r} c_j(n) A(n-j) = 0 \qquad \text{with } A(0) = 1 \text{ and } A(j) = 0 \text{ for } j < 0.$$

The $c_j(n)$ are polynomials with $c_0(n) \in n^2\mathbb{Z}[n]$ and $c_0(n) \neq 0$ for $n > 0$.

# The formal derivative of recurrence sequences

- Suppose $A(n)$ is the unique solution for all $n \geqslant 0$ to

$$\sum_{j=0}^{r} c_j(n) A(n-j) = 0 \qquad \text{with } A(0) = 1 \text{ and } A(j) = 0 \text{ for } j < 0.$$

  The $c_j(n)$ are polynomials with $c_0(n) \in n^2\mathbb{Z}[n]$ and $c_0(n) \neq 0$ for $n > 0$.
- Then the **formal derivative** $A'(n)$ is the unique solution to

$$\sum_{j=0}^{r} c_j(n) A'(n-j) + \sum_{j=0}^{r} c_j'(n) A(n-j) = 0 \qquad \text{with } A'(j) = 0 \text{ for } j \leqslant 0.$$

# The formal derivative of recurrence sequences

- Suppose $A(n)$ is the unique solution for all $n \geqslant 0$ to

$$\sum_{j=0}^{r} c_j(n) A(n-j) = 0 \qquad \text{with } A(0) = 1 \text{ and } A(j) = 0 \text{ for } j < 0.$$

  The $c_j(n)$ are polynomials with $c_0(n) \in n^2\mathbb{Z}[n]$ and $c_0(n) \neq 0$ for $n > 0$.

- Then the **formal derivative** $A'(n)$ is the unique solution to

$$\sum_{j=0}^{r} c_j(n) A'(n-j) + \sum_{j=0}^{r} c_j'(n) A(n-j) = 0 \qquad \text{with } A'(j) = 0 \text{ for } j \leqslant 0.$$

**Note** Let $F(x) = \sum_{n \geqslant 0} A(n) x^n$ and $G(x) = \sum_{n \geqslant 1} A'(n) x^n$.

Then the corresponding differential equation satisfied by $F(x)$ is also solved by $\log(x) F(x) + G(x)$.

# The formal derivative of recurrence sequences: example

- $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ is the unique solution with $A(0) = 1$ to:

$$(n+1)^2 A(n+1) = (11n^2 + 11n + 3) A(n) + n^2 A(n-1)$$

- Then $A'(n)$ is the unique solution with $A'(0) = 0$ to:

$$(n+1)^2 A'(n+1) = (11n^2 + 11n + 3) A'(n) + n^2 A'(n-1)$$
$$- 2(n+1)A(n+1) + 11(2n+1)A(n) + 2nA(n-1)$$

# The formal derivative of recurrence sequences: example

- $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ is the unique solution with $A(0) = 1$ to:

$$(n+1)^2 A(n+1) = (11n^2 + 11n + 3) A(n) + n^2 A(n-1)$$

- Then $A'(n)$ is the unique solution with $A'(0) = 0$ to:

$$\begin{aligned}(n+1)^2 A'(n+1) &= (11n^2 + 11n + 3) A'(n) + n^2 A'(n-1) \\ &\quad - 2(n+1) A(n+1) + 11(2n+1) A(n) + 2n A(n-1)\end{aligned}$$

**EG** $A'(1), A'(2), \ldots = 5,\ \frac{75}{2},\ \frac{1855}{6},\ \frac{10875}{4},\ \frac{299387}{12},\ \frac{943397}{4},\ \frac{63801107}{28},\ \ldots$

# The formal derivative of recurrence sequences: example

- $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ is the unique solution with $A(0) = 1$ to:

$$(n+1)^2 A(n+1) = (11n^2 + 11n + 3) A(n) + n^2 A(n-1)$$

- Then $A'(n)$ is the unique solution with $A'(0) = 0$ to:

$$\begin{aligned}(n+1)^2 A'(n+1) &= (11n^2 + 11n + 3) A'(n) + n^2 A'(n-1) \\ &\quad - 2(n+1) A(n+1) + 11(2n+1) A(n) + 2n A(n-1)\end{aligned}$$

**EG** $A'(1), A'(2), \ldots = 5,\ \frac{75}{2},\ \frac{1855}{6},\ \frac{10875}{4},\ \frac{299387}{12},\ \frac{943397}{4},\ \frac{63801107}{28},\ \ldots$

- Since the interpolation satisfies the continuous version of the recurrence:

$$\begin{aligned}A'(n) &= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k} \Bigg|_{x=n} \\ &= 5 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} (H_n - H_k)\end{aligned}$$

# Approaches to proving Lucas congruences

- From suitable expressions as a **binomial sum**. Gessel '82, McIntosh '92

Apéry numbers: $\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$

Sequence $(\eta)$: $\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$

- From suitable expressions as a **binomial sum**. Gessel '82, McIntosh '92

Apéry numbers: $\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$

Sequence $(\eta)$: $\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$

- From suitable **constant term** expressions. Samol–van Straten '09, Mellit–Vlasenko '16

**THM** Samol, van Straten '09: Suppose the origin is the only interior integral point of the Newton polytope of $P \in \mathbb{Z}[\boldsymbol{x}^{\pm 1}]$.
Then $A(n) = \operatorname{ct}[P(\boldsymbol{x})^n]$ satisfies Lucas congruences.

$$P = \frac{(x+y)(z+1)(x+y+z)(y+z+1)}{xyz}$$

$$\left(1 - \frac{1}{xy(1+z)^5}\right) \frac{(1+x)(1+y)(1+z)^4}{z^3}$$

- From suitable expressions as a **binomial sum**. Gessel '82, McIntosh '92

Apéry numbers: $\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$

Sequence $(\eta)$: $\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$

- From suitable **constant term** expressions. Samol–van Straten '09, Mellit–Vlasenko '16

**THM** Samol, van Straten '09: Suppose the origin is the only interior integral point of the Newton polytope of $P \in \mathbb{Z}[\boldsymbol{x}^{\pm 1}]$.

Then $A(n) = \operatorname{ct}[P(\boldsymbol{x})^n]$ satisfies Lucas congruences.


$$P = \frac{(x+y)(z+1)(x+y+z)(y+z+1)}{xyz}$$

$$\left(1 - \frac{1}{xy(1+z)^5}\right) \frac{(1+x)(1+y)(1+z)^4}{z^3}$$

- From suitable **diagonal** expressions. Rowland–Yassawi '15

For instance, diagonals of $1/Q(\boldsymbol{x})$ for $Q(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$ with $Q(\boldsymbol{x})$ linear in each variable and $Q(\mathbf{0}) = 1$.

# Approaches to proving Lucas congruences

- From suitable expressions as a **binomial sum**. Gessel '82, McIntosh '92

Apéry numbers: $\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$

Sequence $(\eta)$: $\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$

- From suitable **constant term** expressions. Samol–van Straten '09, Mellit–Vlasenko '16

**THM** Samol, van Straten '09: Suppose the origin is the only interior integral point of the Newton polytope of $P \in \mathbb{Z}[\boldsymbol{x}^{\pm 1}]$.

Then $A(n) = \operatorname{ct}[P(\boldsymbol{x})^n]$ satisfies Lucas congruences.

$$P = \frac{(x+y)(z+1)(x+y+z)(y+z+1)}{xyz}$$

$$\left(1 - \frac{1}{xy(1+z)^5}\right) \frac{(1+x)(1+y)(1+z)^4}{z^3}$$

- From suitable **diagonal** expressions. Rowland–Yassawi '15

  For instance, diagonals of $1/Q(\boldsymbol{x})$ for $Q(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$ with $Q(\boldsymbol{x})$ linear in each variable and $Q(\mathbf{0}) = 1$.

- From suitable **modular parametrizations**. Beukers–Tsai–Ye '25

# Lucas congruences in terms of the GF

- Given $F(x) = \sum_{n=0}^{\infty} A(n)x^n$, we write $F_p(x) = \sum_{n=0}^{p-1} A(n)x^n$ for its $p$-**truncation**.

**LEM** $A(n)$ satisfies Lucas congruences modulo $p$
$\iff$ $\frac{1}{F^{p-1}(x)}$ modulo $p$ is a polynomial of degree $< p$.

**proof**

$$A(n) \equiv A(n_0)A(n_1)A(n_2)\cdots \pmod{p}$$

$$\iff \quad F(x) \equiv F_p(x)\, F_p(x^p)F_p(x^{p^2})\cdots \pmod{p}$$

□

# Lucas congruences in terms of the GF

- Given $F(x) = \sum_{n=0}^{\infty} A(n) x^n$, we write $F_p(x) = \sum_{n=0}^{p-1} A(n) x^n$ for its $p$-**truncation**.

**LEM** $A(n)$ satisfies Lucas congruences modulo $p$
$\iff \quad \frac{1}{F^{p-1}(x)}$ modulo $p$ is a polynomial of degree $< p$.

**proof**

$$\begin{aligned} & A(n) \equiv A(n_0) A(n_1) A(n_2) \cdots && \pmod{p} \\ \iff \quad & F(x) \equiv F_p(x)\, F_p(x^p) F_p(x^{p^2}) \cdots && \pmod{p} \\ \iff \quad & F(x) \equiv F_p(x)\, F(x^p) && \pmod{p} \end{aligned}$$

□

# Lucas congruences in terms of the GF

- Given $F(x) = \sum_{n=0}^{\infty} A(n)x^n$, we write $F_p(x) = \sum_{n=0}^{p-1} A(n)x^n$ for its $p$-**truncation**.

**LEM** $A(n)$ satisfies Lucas congruences modulo $p$
$\iff$ $\frac{1}{F^{p-1}(x)}$ modulo $p$ is a polynomial of degree $< p$.

**proof**

$$A(n) \equiv A(n_0)A(n_1)A(n_2)\cdots \pmod{p}$$

$$\iff \quad F(x) \equiv F_p(x)\, F_p(x^p)F_p(x^{p^2})\cdots \pmod{p}$$

$$\iff \quad F(x) \equiv F_p(x)\, F(x^p) \pmod{p}$$

$$\iff \quad F_p(x) \equiv \frac{F(x)}{F(x^p)} \pmod{p}$$

□

# Lucas congruences in terms of the GF

- Given $F(x) = \sum_{n=0}^{\infty} A(n)x^n$, we write $F_p(x) = \sum_{n=0}^{p-1} A(n)x^n$ for its $p$-**truncation**.

**LEM** $A(n)$ satisfies Lucas congruences modulo $p$

$\iff \quad \frac{1}{F^{p-1}(x)}$ modulo $p$ is a polynomial of degree $< p$.

**proof**

$$A(n) \equiv A(n_0)A(n_1)A(n_2)\cdots \pmod{p}$$

$$\iff \quad F(x) \equiv F_p(x)\, F_p(x^p)F_p(x^{p^2})\cdots \pmod{p}$$

$$\iff \quad F(x) \equiv F_p(x)\, F(x^p) \pmod{p}$$

$$\iff \quad F_p(x) \equiv \frac{F(x)}{F(x^p)} \pmod{p}$$

(by little Fermat) $\equiv \frac{F(x)}{F^p(x)}$

□

# Lucas congruences in terms of the GF

- Given $F(x) = \sum_{n=0}^{\infty} A(n)x^n$, we write $F_p(x) = \sum_{n=0}^{p-1} A(n)x^n$ for its $p$-**truncation**.

**LEM** $A(n)$ satisfies Lucas congruences modulo $p$
$\iff$ $\frac{1}{F^{p-1}(x)}$ modulo $p$ is a polynomial of degree $< p$.

**proof**

$$A(n) \equiv A(n_0)A(n_1)A(n_2)\cdots \pmod{p}$$

$$\iff \quad F(x) \equiv F_p(x)\, F_p(x^p)F_p(x^{p^2})\cdots \pmod{p}$$

$$\iff \quad F(x) \equiv F_p(x)\, F(x^p) \pmod{p}$$

$$\iff \quad F_p(x) \equiv \frac{F(x)}{F(x^p)} \pmod{p}$$

(by little Fermat) $$\equiv \frac{F(x)}{F^p(x)} = \frac{1}{F^{p-1}(x)}$$

□

# Lucas congruences in terms of the GF

- Given $F(x) = \sum_{n=0}^{\infty} A(n)x^n$, we write $F_p(x) = \sum_{n=0}^{p-1} A(n)x^n$ for its $p$-**truncation**.

**LEM** $A(n)$ satisfies Lucas congruences modulo $p$
$\iff \quad \frac{1}{F^{p-1}(x)}$ modulo $p$ is a polynomial of degree $< p$.

**proof**

$$A(n) \equiv A(n_0)A(n_1)A(n_2)\cdots \pmod{p}$$

$$\iff \quad F(x) \equiv F_p(x)\, F_p(x^p)F_p(x^{p^2})\cdots \pmod{p}$$

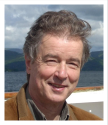
$$\iff \quad F(x) \equiv F_p(x)\, F(x^p) \pmod{p}$$

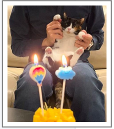
$$\iff \quad F_p(x) \equiv \frac{F(x)}{F(x^p)} \pmod{p}$$

(by little Fermat) $\equiv \frac{F(x)}{F^p(x)} = \frac{1}{F^{p-1}(x)}$

Since the first $p$ coefficients of ... always match, the final congruence is equivalent to the RHS being a polynomial of degree $\leqslant p-1$. □

# Lucas congruences via modular forms





- Suppose $F(x) = \sum_{n=0}^{\infty} A(n) x^n$ has **modular parametrization**:

  $F(x)$ is a modular form for some modular function $x(\tau)$.

**THM** Beukers–Tsai–Ye '25

Suppose that:

- $x(\tau) = q + q^2\mathbb{Z}[[q]]$ with $q = e^{2\pi i\tau}$ is a **Hauptmodul** for $\Gamma = \Gamma_0(N)$ (or Atkin–Lehner extension).
- $F(x(\tau)) = 1 + q\mathbb{Z}[[q]]$ is a weight $2$ modular form for $\Gamma$.
- $F(x(\tau))$ has a unique zero at $[\tau_0]$ of order $\leqslant 1$, where $[\tau_0]$ is the (unique) pole of $x(\tau)$.

Then $A(n)$ satisfies the Lucas congruences for all primes $p$.

# Lucas congruences via modular forms

- Suppose $F(x) = \sum_{n=0}^{\infty} A(n) x^n$ has **modular parametrization**:

  $F(x)$ is a modular form for some modular function $x(\tau)$.

**THM** Beukers–Tsai–Ye '25

Suppose that:

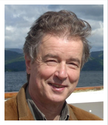
- $x(\tau) = q + q^2\mathbb{Z}[[q]]$ with $q = \mathrm{e}^{2\pi i\tau}$ is a **Hauptmodul** for $\Gamma = \Gamma_0(N)$ (or Atkin–Lehner extension).
- $F(x(\tau)) = 1 + q\mathbb{Z}[[q]]$ is a weight $2$ modular form for $\Gamma$.
- $F(x(\tau))$ has a unique zero at $[\tau_0]$ of order $\leqslant 1$, where $[\tau_0]$ is the (unique) pole of $x(\tau)$.

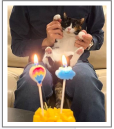
Then $A(n)$ satisfies the Lucas congruences for all primes $p$.

**proof** Suppose $E(\tau)$ is a modular form for $\Gamma$ with weight $2(p-1)$ such that $E(\tau) \equiv 1 \pmod{p}$.

□

# Lucas congruences via modular forms





- Suppose $F(x) = \sum_{n=0}^{\infty} A(n) x^n$ has **modular parametrization**:

  $F(x)$ is a modular form for some modular function $x(\tau)$.

**THM** Beukers–Tsai–Ye '25

Suppose that:

- $x(\tau) = q + q^2\mathbb{Z}[[q]]$ with $q = e^{2\pi i\tau}$ is a **Hauptmodul** for $\Gamma = \Gamma_0(N)$ (or Atkin–Lehner extension).
- $F(x(\tau)) = 1 + q\mathbb{Z}[[q]]$ is a weight $2$ modular form for $\Gamma$.
- $F(x(\tau))$ has a unique zero at $[\tau_0]$ of order $\leqslant 1$, where $[\tau_0]$ is the (unique) pole of $x(\tau)$.

Then $A(n)$ satisfies the Lucas congruences for all primes $p$.

**proof** Suppose $E(\tau)$ is a modular form for $\Gamma$ with weight $2(p-1)$ such that $E(\tau) \equiv 1 \pmod{p}$. Then

$$\frac{1}{F^{p-1}(x)} \equiv \qquad\qquad\qquad\qquad \pmod{p}.$$

□

# Lucas congruences via modular forms

- Suppose $F(x) = \sum_{n=0}^{\infty} A(n) x^n$ has **modular parametrization**:
  $F(x)$ is a modular form for some modular function $x(\tau)$.

**THM** Beukers–Tsai–Ye '25

Suppose that:

- $x(\tau) = q + q^2 \mathbb{Z}[[q]]$ with $q = e^{2\pi i \tau}$ is a **Hauptmodul** for $\Gamma = \Gamma_0(N)$ (or Atkin–Lehner extension).
- $F(x(\tau)) = 1 + q\mathbb{Z}[[q]]$ is a weight $2$ modular form for $\Gamma$.
- $F(x(\tau))$ has a unique zero at $[\tau_0]$ of order $\leqslant 1$, where $[\tau_0]$ is the (unique) pole of $x(\tau)$.

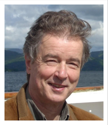
Then $A(n)$ satisfies the Lucas congruences for all primes $p$.

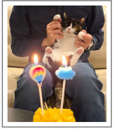
**proof** Suppose $E(\tau)$ is a modular form for $\Gamma$ with weight $2(p-1)$ such that $E(\tau) \equiv 1 \pmod{p}$. Then

$$\frac{1}{F^{p-1}(x)} \equiv \frac{E(\tau)}{F^{p-1}(x)} \pmod{p}.$$

is a modular function with a unique pole at $[\tau_0]$ of order $\leqslant p-1$. □

# Lucas congruences via modular forms





- Suppose $F(x) = \sum_{n=0}^{\infty} A(n) x^n$ has **modular parametrization**:

  $F(x)$ is a modular form for some modular function $x(\tau)$.

**THM** Beukers–Tsai–Ye '25

Suppose that:

- $x(\tau) = q + q^2 \mathbb{Z}[[q]]$ with $q = e^{2\pi i \tau}$ is a **Hauptmodul** for $\Gamma = \Gamma_0(N)$ (or Atkin–Lehner extension).
- $F(x(\tau)) = 1 + q\mathbb{Z}[[q]]$ is a weight $2$ modular form for $\Gamma$.
- $F(x(\tau))$ has a unique zero at $[\tau_0]$ of order $\leqslant 1$, where $[\tau_0]$ is the (unique) pole of $x(\tau)$.

Then $A(n)$ satisfies the Lucas congruences for all primes $p$.

**proof** Suppose $E(\tau)$ is a modular form for $\Gamma$ with weight $2(p-1)$ such that $E(\tau) \equiv 1 \pmod{p}$. Then

$$\frac{1}{F^{p-1}(x)} \equiv \frac{E(\tau)}{F^{p-1}(x)} = \operatorname{poly}(x) \pmod{p}.$$

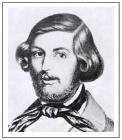
is a modular function with a unique pole at $[\tau_0]$ of order $\leqslant p-1$. □

# Lucas congruences via modular forms, cont'd

- Needed: weight $2(p-1)$ modular form $E(\tau)$ for $\Gamma$ with $E(\tau) \equiv 1 \pmod{p}$.

**EG** The normalized **Eisenstein series**

$$E_k(\tau) = 1 + \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}$$



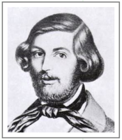
is a modular form for $\Gamma_0(1)$ of even weight $k \geqslant 2$.

Since $1/B_{p-1} \equiv 0 \pmod{p}$, we have $E_{p-1}(\tau) \equiv 1 \pmod{p}$.

# Lucas congruences via modular forms, cont'd

- Needed: weight $2(p-1)$ modular form $E(\tau)$ for $\Gamma$ with $E(\tau) \equiv 1 \pmod{p}$.

**EG** The normalized **Eisenstein series**



$$E_k(\tau) = 1 + \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}$$

is a modular form for $\Gamma_0(1)$ of even weight $k \geqslant 2$.

Since $1/B_{p-1} \equiv 0 \pmod{p}$, we have $E_{p-1}(\tau) \equiv 1 \pmod{p}$.

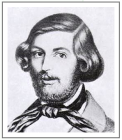
- If $p \geqslant 5$ and $\Gamma = \Gamma_0(N)$, we can select:

  $E(\tau) := E_{p-1}(\tau)^2$

# Lucas congruences via modular forms, cont'd

- Needed: weight $2(p-1)$ modular form $E(\tau)$ for $\Gamma$ with $E(\tau) \equiv 1 \pmod{p}$.

**EG** The normalized **Eisenstein series**



$$E_k(\tau) = 1 + \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}$$

is a modular form for $\Gamma_0(1)$ of even weight $k \geqslant 2$.

Since $1/B_{p-1} \equiv 0 \pmod{p}$, we have $E_{p-1}(\tau) \equiv 1 \pmod{p}$.

- If $p \geqslant 5$ and $\Gamma = \Gamma_0(N)$, we can select:

  $E(\tau) := E_{p-1}(\tau)^2$

- If $p \geqslant 5$ and $\Gamma$ is $\Gamma_0(N)$ extended by $\tau \to -\frac{1}{N\tau}$:

  $E(\tau) := \frac{1}{2} \left[ E_{p-1}(\tau)^2 + N^{p-1} E_{p-1}(N\tau)^2 \right]$

# Time?

Bonus material:

**Lucas** and **Gessel–Lucas** congruences are natural from the point of view of **congruence automata**

# Sporadic sequences mod $p^r$ are automatic

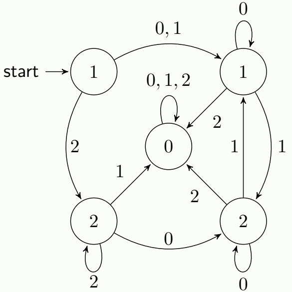
**THM** Rowland, Yassawi '15 — If an integer sequence $A(n)$ is the diagonal of $F(\boldsymbol{x}) \in \mathbb{Z}(\boldsymbol{x})$, then the reductions $A(n) \pmod{p^r}$ are $p$-**automatic**.

Constructive proof of results by Denef and Lipshitz '87.

**THM** Rowland, Yassawi '15: If an integer sequence $A(n)$ is the diagonal of $F(\boldsymbol{x}) \in \mathbb{Z}(\boldsymbol{x})$, then the reductions $A(n) \pmod{p^r}$ are $p$-**automatic**.

Constructive proof of results by Denef and Lipshitz '87.

**EG** Catalan numbers $C(n)$ modulo $3$:




$$C(35) = 3{,}116{,}285{,}494{,}907{,}301{,}262$$
$$\equiv 1 \pmod{3}$$

Instead via automaton:

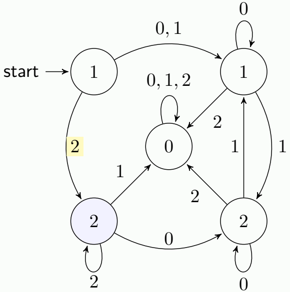
$35 = 1\ 0\ 2\ 2$ in base $3$

# Sporadic sequences mod $p^r$ are automatic

**THM** Rowland, Yassawi '15: If an integer sequence $A(n)$ is the diagonal of $F(\boldsymbol{x}) \in \mathbb{Z}(\boldsymbol{x})$, then the reductions $A(n) \pmod{p^r}$ are $p$-**automatic**.

Constructive proof of results by Denef and Lipshitz '87.

**EG** Catalan numbers $C(n)$ modulo $3$:




$$C(35) = 3{,}116{,}285{,}494{,}907{,}301{,}262$$
$$\equiv 1 \pmod 3$$

Instead via automaton:

$35 = 1\ 0\ 2\ 2$ in base $3$

$C(2)$

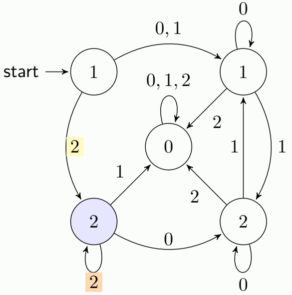
$C(2) \equiv 2$

# Sporadic sequences mod $p^r$ are automatic

**THM** Rowland, Yassawi '15 — If an integer sequence $A(n)$ is the diagonal of $F(\boldsymbol{x}) \in \mathbb{Z}(\boldsymbol{x})$, then the reductions $A(n) \pmod{p^r}$ are $p$-**automatic**.

Constructive proof of results by Denef and Lipshitz '87.

**EG** Catalan numbers $C(n)$ modulo $3$:




$$C(35) = 3{,}116{,}285{,}494{,}907{,}301{,}262$$
$$\equiv 1 \pmod 3)$$

Instead via automaton:

$35 = 1\ 0\ 2\ 2$ in base $3$

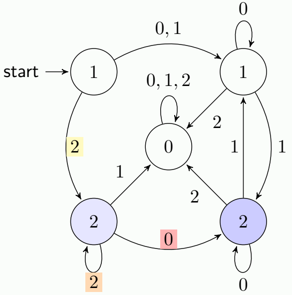
$C(2)$ $\qquad C(2) \equiv 2$

$C(8)$ $\qquad C(2\ 2) \equiv 2$

**THM** Rowland, Yassawi '15: If an integer sequence $A(n)$ is the diagonal of $F(\boldsymbol{x}) \in \mathbb{Z}(\boldsymbol{x})$, then the reductions $A(n) \pmod{p^r}$ are $p$-**automatic**.

Constructive proof of results by Denef and Lipshitz '87.

**EG** Catalan numbers $C(n)$ modulo $3$:




$$C(35) = 3{,}116{,}285{,}494{,}907{,}301{,}262$$
$$\equiv 1 \pmod 3$$

Instead via automaton:

$35 = 1\ 0\ 2\ 2$ in base 3

$C(2)$ $\quad C(2) \equiv 2$

$C(8)$ $\quad C(22) \equiv 2$

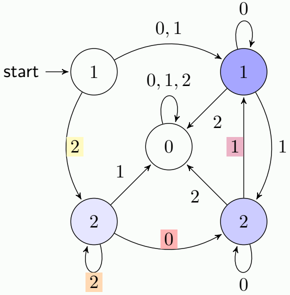
$C(022) \equiv 2$

# Sporadic sequences mod $p^r$ are automatic


**THM** Rowland, Yassawi '15: If an integer sequence $A(n)$ is the diagonal of $F(\boldsymbol{x}) \in \mathbb{Z}(\boldsymbol{x})$, then the reductions $A(n) \pmod{p^r}$ are $p$-**automatic**.

Constructive proof of results by Denef and Lipshitz '87.

**EG** Catalan numbers $C(n)$ modulo 3:




$$C(35) = 3{,}116{,}285{,}494{,}907{,}301{,}262$$
$$\equiv 1 \pmod 3$$

Instead via automaton:

$35 = 1\,0\,2\,2$ in base 3

$C(2)$ $\quad C(2) \equiv 2$

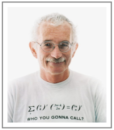
$C(8)$ $\quad C(22) \equiv 2$

$C(022) \equiv 2$

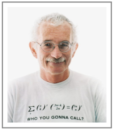
$C(35)$ $\quad C(1022) \equiv 1$

# Linear congruence schemes

- The Catalan numbers $C(n)$ modulo $3$ can be described:
  - by an automaton with $4$ states (plus a zero state)
  - by a **linear** $3$**-scheme** with $2$ states (Rowland–Zeilberger '14)

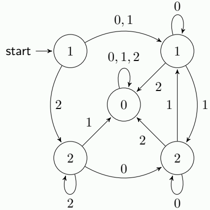


- The Catalan numbers $C(n)$ modulo $3$ can be described:
  - by an automaton with $4$ states (plus a zero state)
  - by a **linear** $3$**-scheme** with $2$ states (Rowland–Zeilberger '14)




**EG** mod $3$

**automatic 3-scheme**

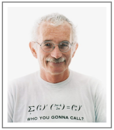



$$\begin{aligned} A_0(3n) &= A_1(n) & A_2(3n) &= A_3(n) \\ A_0(3n+1) &= A_1(n) & A_2(3n+1) &= 0 \\ A_0(3n+2) &= A_2(n) & A_2(3n+2) &= A_2(n) \\ A_1(3n) &= A_1(n) & A_3(3n) &= A_3(n) \\ A_1(3n+1) &= A_3(n) & A_3(3n+1) &= A_1(n) \\ A_1(3n+2) &= 0 & A_3(3n+2) &= 0 \end{aligned}$$

Initial conditions:
$A_0(0) = A_1(0) = 1, \quad A_2(0) = A_3(0) = 2$

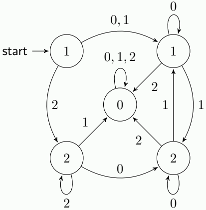
# Linear congruence schemes

- The Catalan numbers $C(n)$ modulo $3$ can be described:
  - by an automaton with $4$ states (plus a zero state)
  - by a **linear $3$-scheme** with $2$ states (Rowland–Zeilberger '14)




**EG** mod $3$

automatic 3-scheme




$$
\begin{aligned}
A_0(3n) &= A_1(n) & A_2(3n) &= A_3(n)\\
A_0(3n+1) &= A_1(n) & A_2(3n+1) &= 0\\
A_0(3n+2) &= A_2(n) & A_2(3n+2) &= A_2(n)\\
A_1(3n) &= A_1(n) & A_3(3n) &= A_3(n)\\
A_1(3n+1) &= A_3(n) & A_3(3n+1) &= A_1(n)\\
A_1(3n+2) &= 0 & A_3(3n+2) &= 0
\end{aligned}
$$

Initial conditions:
$A_0(0) = A_1(0) = 1, \quad A_2(0) = A_3(0) = 2$

**EG** mod $3$

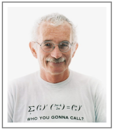
linear 3-scheme

$$
\begin{aligned}
A_0(3n) &= A_1(n) & A_1(3n) &= A_1(n)\\
A_0(3n+1) &= A_1(n) & A_1(3n+1) &= 2A_1(n)\\
A_0(3n+2) &= A_0(n) + A_1(n) & A_1(3n+2) &= 0
\end{aligned}
$$

Initial conditions: $A_0(0) = A_1(0) = 1$

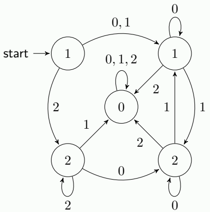
# Linear congruence schemes

- The Catalan numbers $C(n)$ modulo $3$ can be described:
  - by an automaton with $4$ states (plus a zero state)
  - by a **linear $3$-scheme** with $2$ states (Rowland–Zeilberger '14)

**EG** mod $3$

automatic 3-scheme




$$
\begin{aligned}
A_0(3n) &= A_1(n) & A_2(3n) &= A_3(n)\\
A_0(3n+1) &= A_1(n) & A_2(3n+1) &= 0\\
A_0(3n+2) &= A_2(n) & A_2(3n+2) &= A_2(n)\\
A_1(3n) &= A_1(n) & A_3(3n) &= A_3(n)\\
A_1(3n+1) &= A_3(n) & A_3(3n+1) &= A_1(n)\\
A_1(3n+2) &= 0 & A_3(3n+2) &= 0
\end{aligned}
$$

Initial conditions:

$A_0(0) = A_1(0) = 1, \quad A_2(0) = A_3(0) = 2$

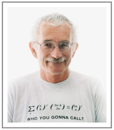
**EG** mod $3$

linear 3-scheme

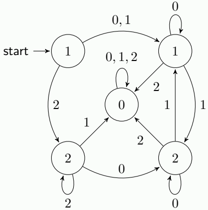
$$
\begin{aligned}
A_0(3n) &= A_1(n) & A_1(3n) &= A_1(n)\\
A_0(3n+1) &= A_1(n) & A_1(3n+1) &= 2A_1(n)\\
A_0(3n+2) &= A_0(n) + A_1(n) & A_1(3n+2) &= 0
\end{aligned}
$$

Initial conditions: $A_0(0) = A_1(0) = 1$

- The Catalan numbers $C(n)$ modulo $3$ can be described:
  - by an automaton with $4$ states (plus a zero state)
  - by a **linear** $3$**-scheme** with $2$ states (Rowland–Zeilberger '14)




**EG** mod $3$ — automatic 3-scheme




$$\begin{aligned} A_0(3n) &= A_1(n) & A_2(3n) &= A_3(n)\\ A_0(3n+1) &= A_1(n) & A_2(3n+1) &= 0\\ A_0(3n+2) &= A_2(n) & A_2(3n+2) &= A_2(n)\\ A_1(3n) &= A_1(n) & A_3(3n) &= A_3(n)\\ A_1(3n+1) &= A_3(n) & A_3(3n+1) &= A_1(n)\\ A_1(3n+2) &= 0 & A_3(3n+2) &= 0 \end{aligned}$$

Initial conditions:

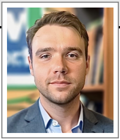
$$A_0(0) = A_1(0) = 1, \quad A_2(0) = A_3(0) = 2$$

**EG** mod $3$ — linear 3-scheme

$$\begin{aligned} A_0(3n) &= A_1(n) & A_1(3n) &= A_1(n)\\ A_0(3n+1) &= A_1(n) & A_1(3n+1) &= 2A_1(n)\\ A_0(3n+2) &= A_0(n)+A_1(n) & A_1(3n+2) &= 0 \end{aligned}$$

Initial conditions: $A_0(0) = A_1(0) = 1$

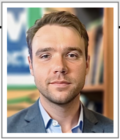
# Few-state linear $p$-schemes



**Lucas congruences:**

$$A(pn+k) \equiv A(k)A(n) \pmod{p}$$

**PROP** Henningsen S '22
$A(n) \pmod{p}$ satisfies a single-state linear $p$-scheme (and $A(0)=1$).
$\iff$ $A(n)$ satisfies Lucas congruences modulo $p$.



**Lucas congruences:**

$$A(pn+k) \equiv A(k)A(n) \pmod{p}$$

**PROP** Henningsen S '22 $A(n) \pmod{p}$ satisfies a single-state linear $p$-scheme (and $A(0)=1$).
$\iff$ $A(n)$ satisfies Lucas congruences modulo $p$.

**Gessel–Lucas congruences:**

$$A(pn+k) \equiv A(k)A(n) + pnA'(k)A(n) \pmod{p^2}$$

**Note** Gessel–Lucas congruences yield explicit $2$-state linear $p$-schemes.

# Conclusions

**THM** S '24 The known sporadic sequences satisfy the **Gessel–Lucas congruences**

$$A(pn+k) \equiv A(k)A(n) + pnA'(k)A(n) \pmod{p^2}.$$

- Lucas congruences correspond to single-state linear $p$-schemes.
  Gessel–Lucas congruences are instances of $2$-state linear $p$-schemes.
- It would be of interest to study **few-state $p$-schemes** systematically:
  - What kind of "generalized Lucas congruences" does one get?
  - Which sequences satisfy such congruences? (mod $p$, mod $p^2$?)

  Partial results by Henningsen–S ('22) for certain constant term sequences.
- Are there interesting $q$-**analogs**?
  - $q$-Lucas congruences have been studied. Olive '65, Désarménien '82
  - For $k = 0$, we get $A(pn) \equiv A(n) \pmod{p^2}$. **(Supercongruences!)**
    $q$-analogs known for some sporadic sequences. S '19, Gorodetsky '19

# THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

**Frits Beukers, Wei-Lun Tsai, Dongxi Ye**
*Lucas congruences using modular forms*
Bulletin of the London Mathematical Society, Vol. 57, 2025, p. 69-78

**Joel Henningsen, Armin Straub**
*Generalized Lucas congruences and linear $p$-schemes*
Advances in Applied Mathematics, Vol. 141, 2022, p. 1-20, #102409

**Armin Straub**
*Gessel-Lucas congruences for sporadic sequences*
Monatshefte für Mathematik, Vol. 203, 2024, p. 883–898