Numbers à la Apéry and their remarkable properties

Quebec-Vermont Number Theory Seminar

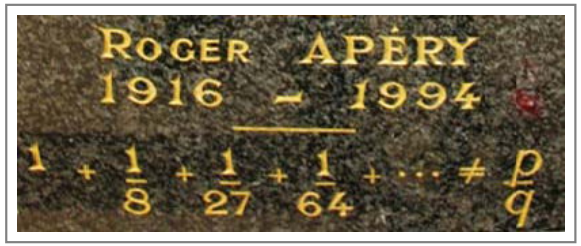
Armin Straub

October 9, 2025

University of South Alabama

CONJ $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

- Apéry (1978): $\zeta(3)$ is irrational
- Open: $\zeta(5)$ is irrational
- Open: $\zeta(3)$ is transcendental
- Open: $\zeta(3)/\pi^3$ is irrational



Slides available at:
`http://arminstraub.com/talks`

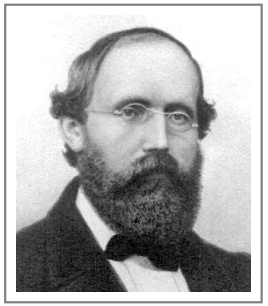
Rough outline

- Introducing Apéry-like numbers
  - they are integer solutions to certain three-term recurrences
  - are all of them known?
- Apéry-like numbers have interesting properties
  - connection to modular forms
  - special p-adic properties
  - multivariate extensions
  - polynomial analogs (skipped today)
- A walk down memory lane: running into Apéry-like numbers
  - planar random walks
  - series for $1/\pi$
  - positivity of rational functions
  - counting points on algebraic varieties (skipped today)
  - ...

The Riemann zeta function

- The **Riemann zeta function** is the analytic continuation of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$



- Its zeros and values are fundamental, yet mysterious to this day.

CONJ RH If $\zeta(s) = 0$ then $s \in \{-2, -4, \ldots\}$ or $\operatorname{Re}(s) = \frac{1}{2}$.

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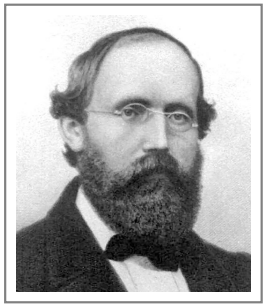

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$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \ldots, \quad \zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2(2n)!}$$

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The values $\zeta(3), \zeta(5), \zeta(7), \ldots$ are all transcendental.

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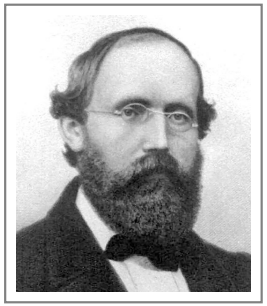

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$\zeta(3)$ is irrational.

Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

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satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

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Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

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> *After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence satisfies the recurrence.*
>
> **Alfred van der Poorten** — *A proof that Euler missed...* **(1979)**

Zagier's search and Apéry-like numbers

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$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \qquad (a,b,c) = (11,3,-1).$$

Q Beukers

Are there other tuples (a,b,c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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- Apart from degenerate cases, Zagier found 6 sporadic integer solutions:

*	$C_*(n)$
A	$\sum_{k=0}^{n} \binom{n}{k}^3$
B	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$
C	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$

*	$C_*(n)$
D	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$
E	$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$
F	$\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$

Almkvist–Zudilin's search for sporadic sequences of order 3

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - cn^3 u_{n-1}$$

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Kauers–Zeilberger diagonal
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Modularity of Apéry-like numbers

- Beukers ('87) observed that the Apéry numbers $1, 5, 73, 1145, \ldots$

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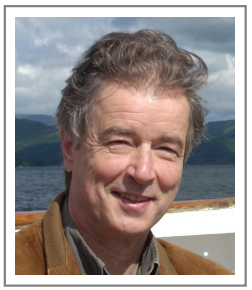

satisfy:

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}$$

$$1 + 5q + 13q^2 + 23q^3 + O(q^4) \qquad q - 12q^2 + 66q^3 + O(q^4)$$

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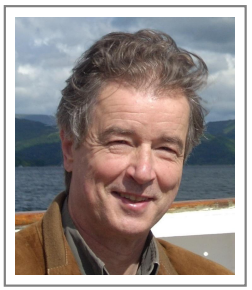

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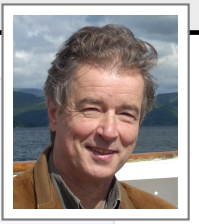
FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

Apéry numbers have remarkable properties



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$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\qquad$ $q = e^{2\pi i\tau}$

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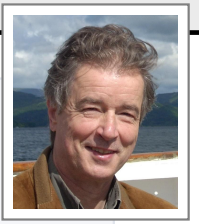

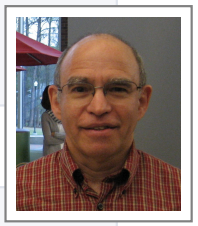

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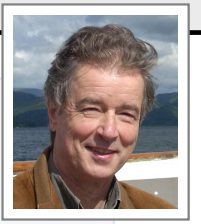
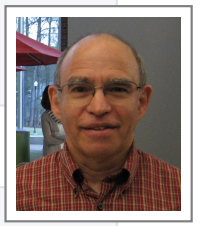


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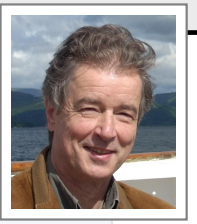
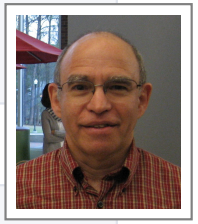


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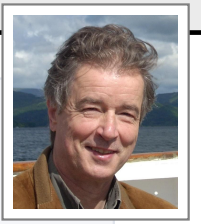
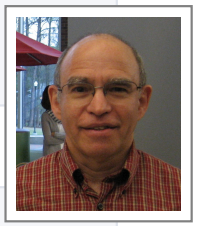
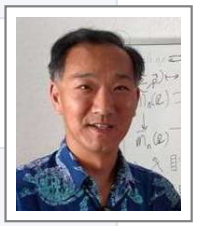
THM Zagier '16

$$A\left(-\frac{1}{2}\right) = \frac{16}{\pi^2} L(f, 2)$$


- These extend to **all known** sporadic (Apéry-like) numbers!!!??

! = proven
? = partially known

Short random walks

based on joint work(s) with:

Jon Borwein
(U. Newcastle, AU)

Dirk Nuyens
(K.U.Leuven, BE)

James Wan
(SUTD, SG)

Wadim Zudilin
(Radboud U., NL)

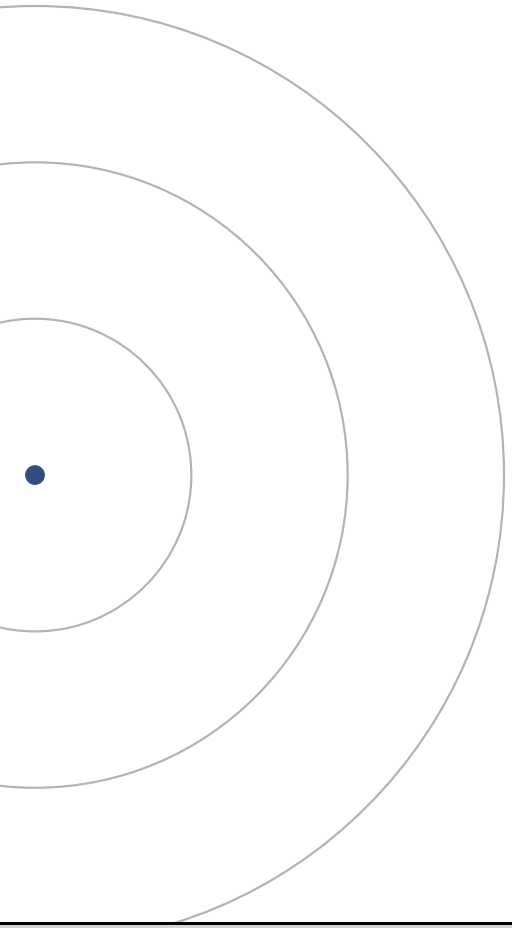
Random walks in the plane

n steps in the plane
(length 1, random direction)

Q What is the distance traveled in n steps?

$p_n(x)$ probability density

$W_n(s)$ sth moment



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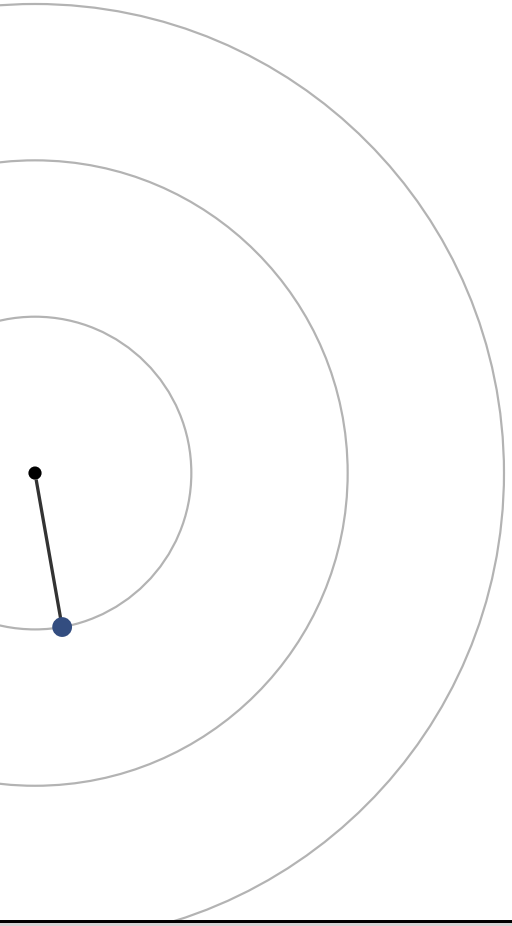

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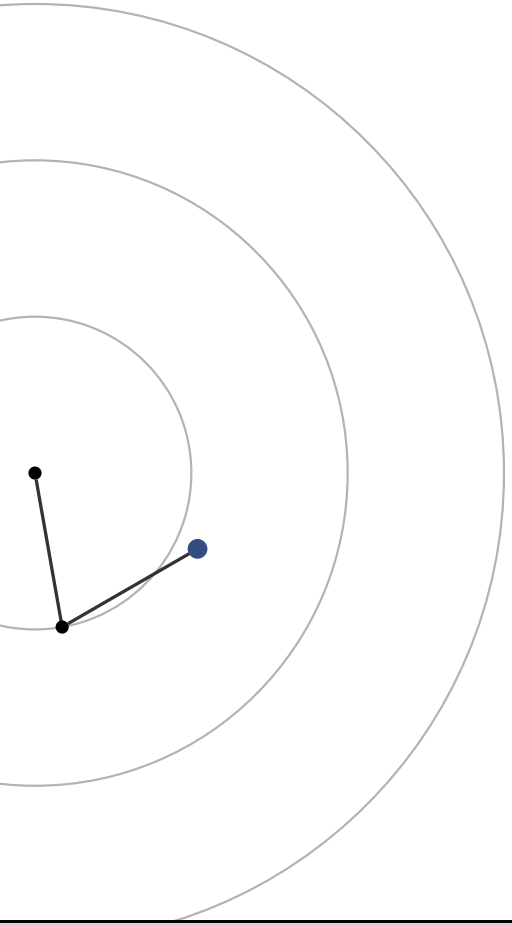

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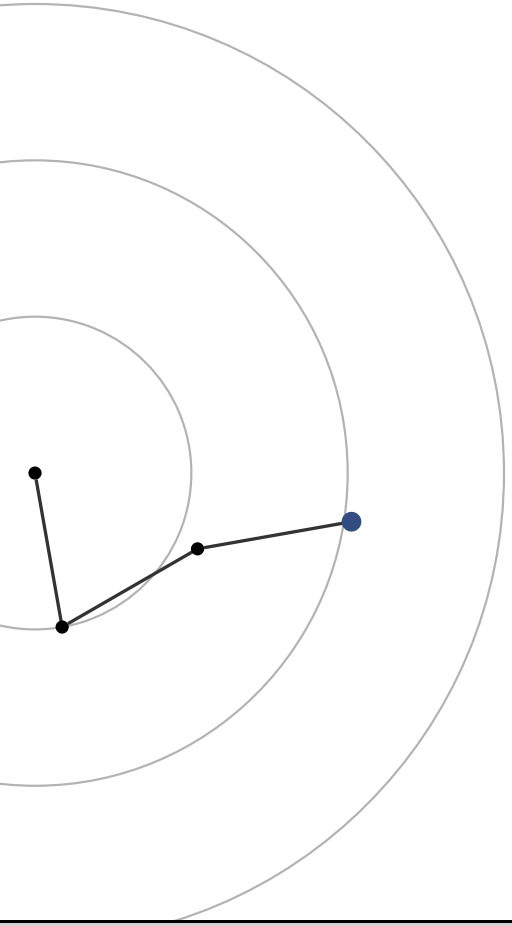

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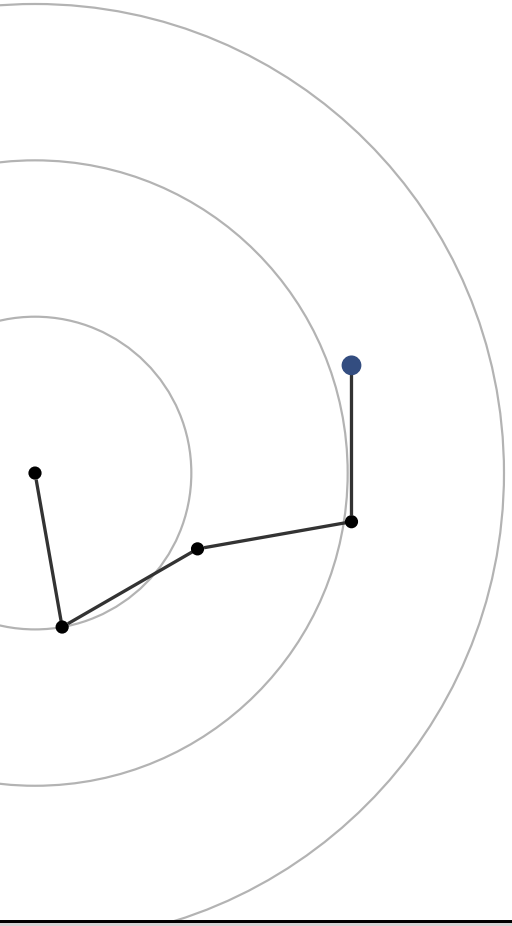

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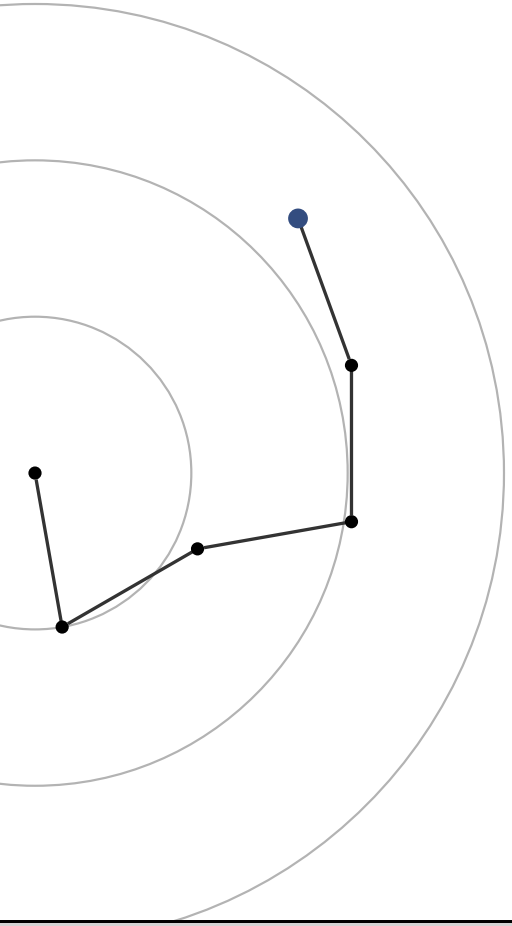

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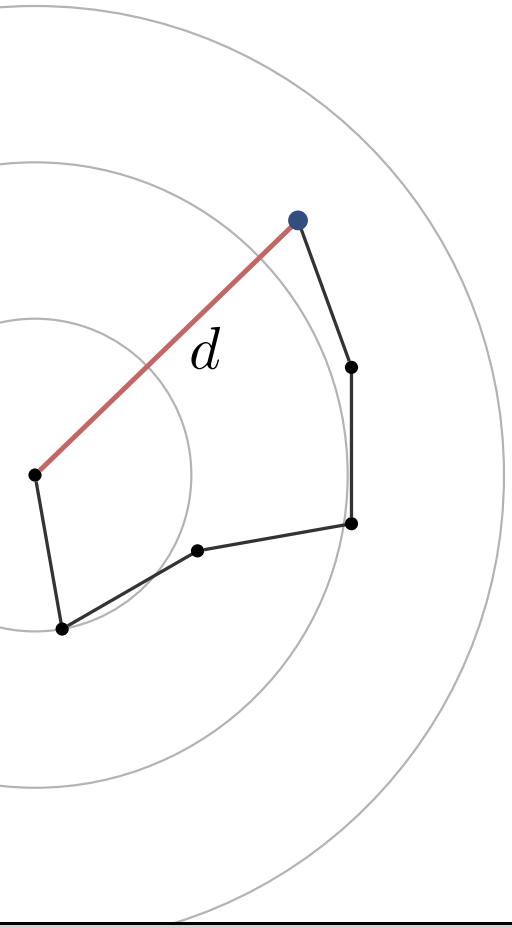
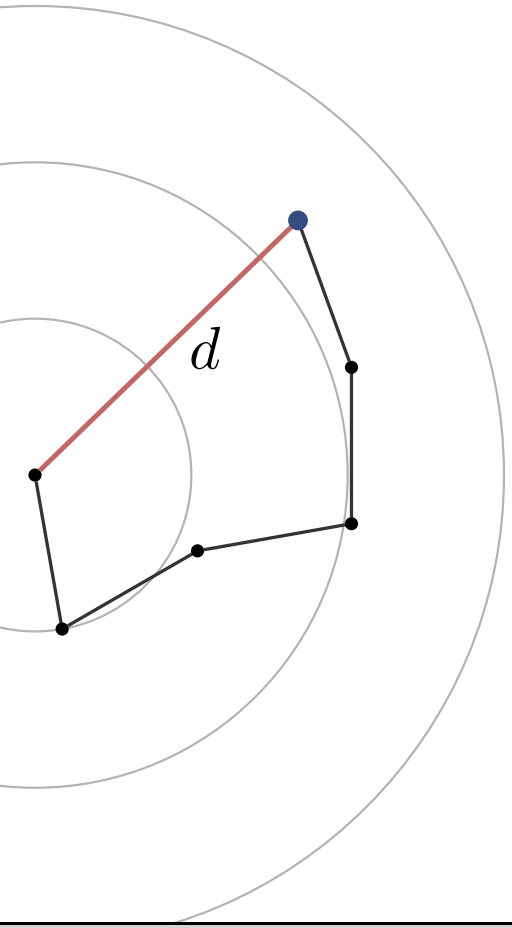


Random walks in the plane

n steps in the plane
(length 1, random direction)



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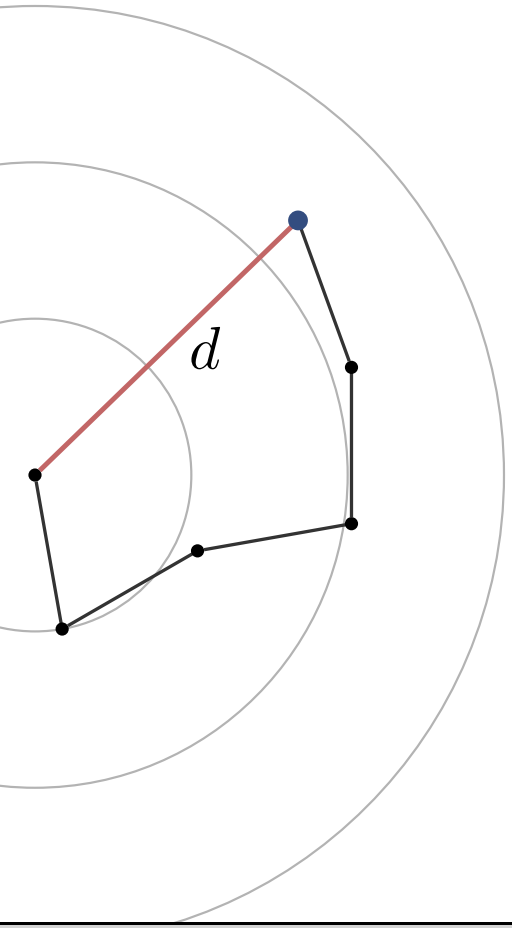

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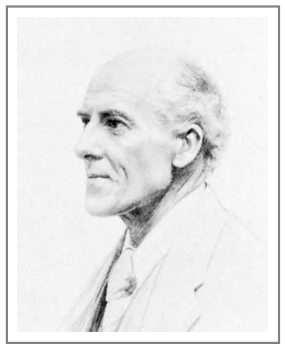

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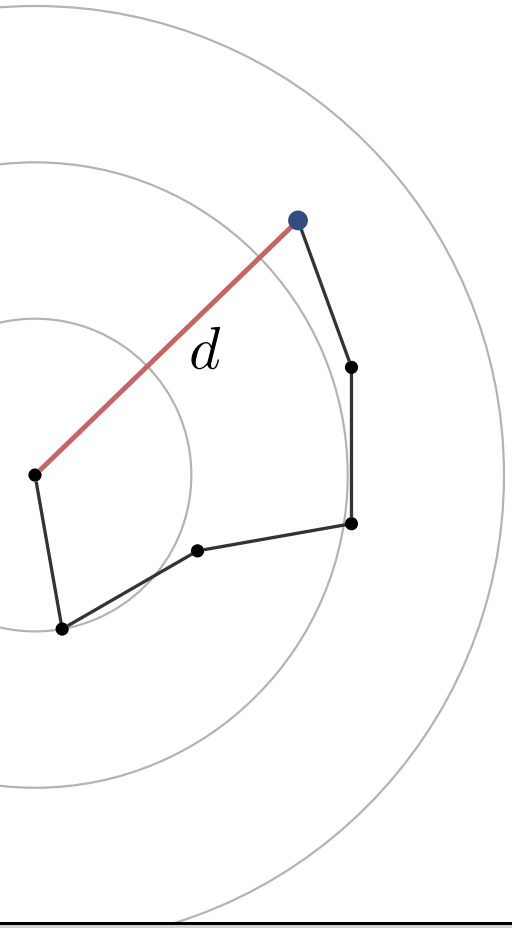

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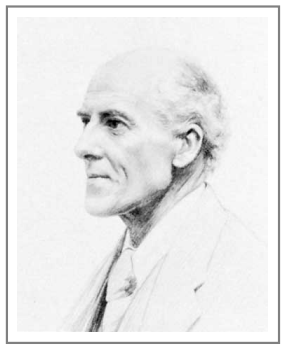

THM Rayleigh, 1905

$$p_n(x) \approx \frac{2x}{n} e^{-x^2/n} \qquad \text{for large } n$$

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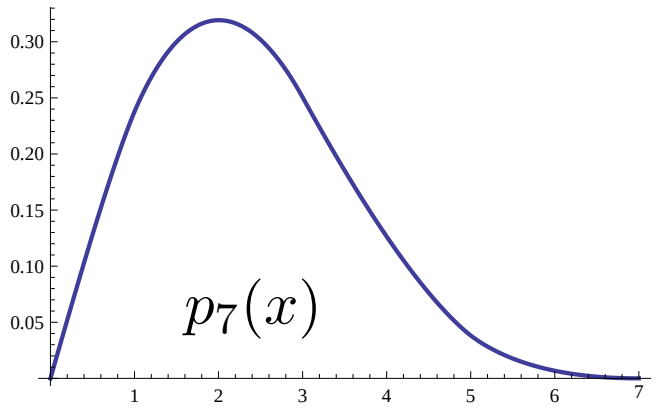

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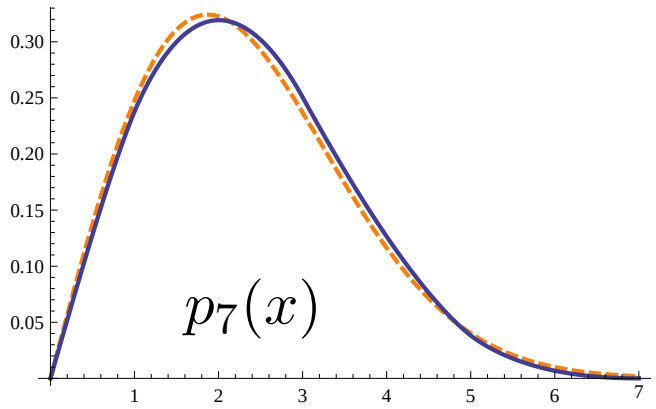

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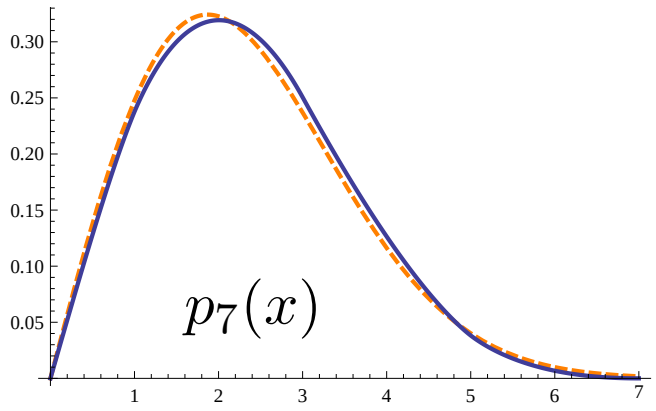

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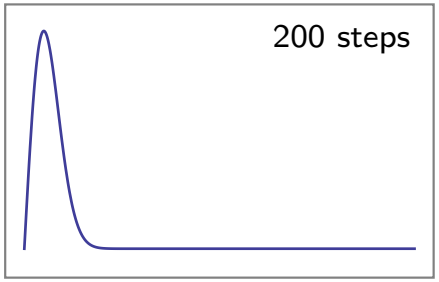
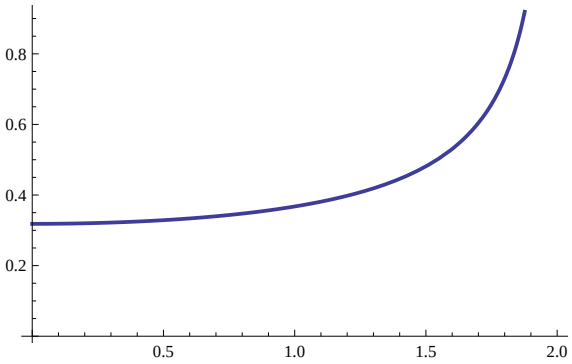
The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!

Karl Pearson, *1905*

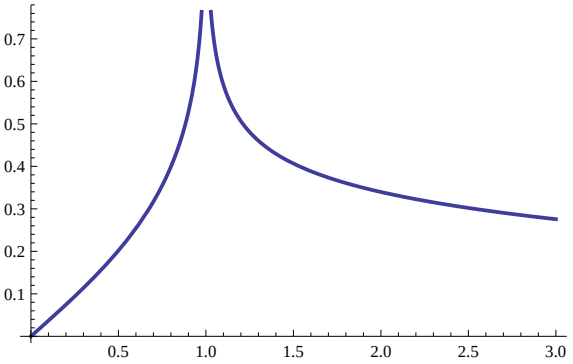


Densities of short walks

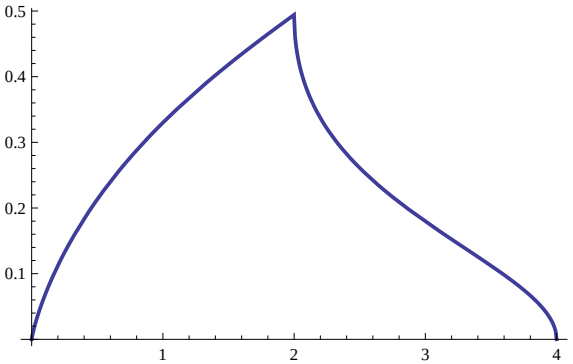
p_2



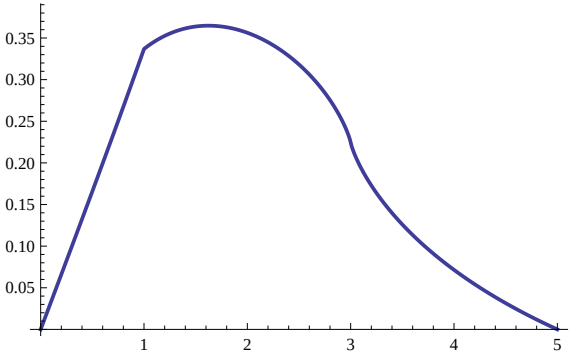
p_3



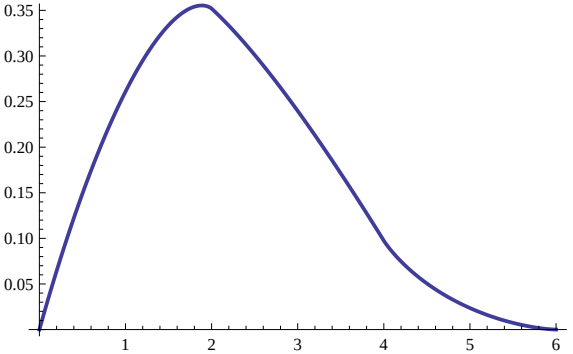
p_4



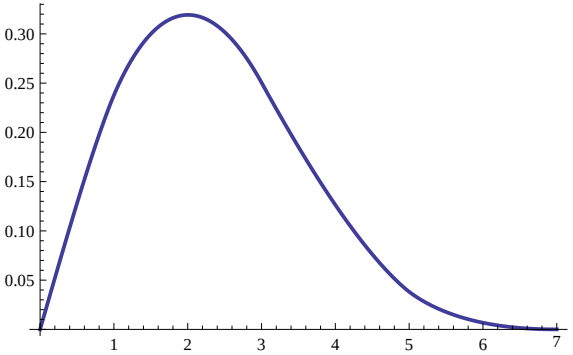
p_5



p_6



p_7



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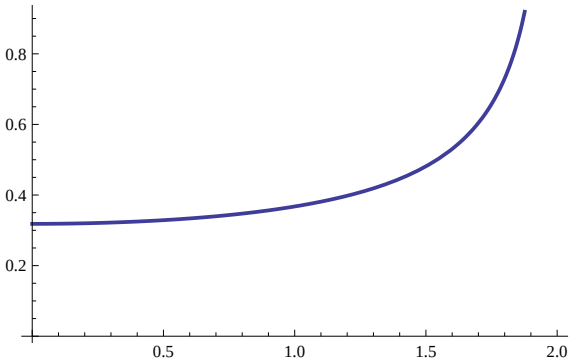

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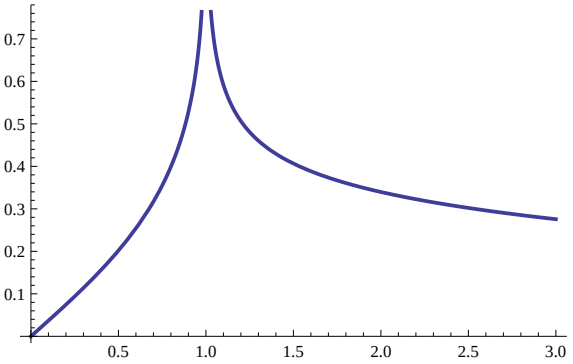

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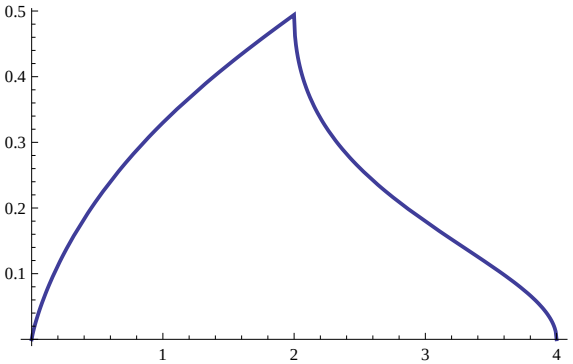

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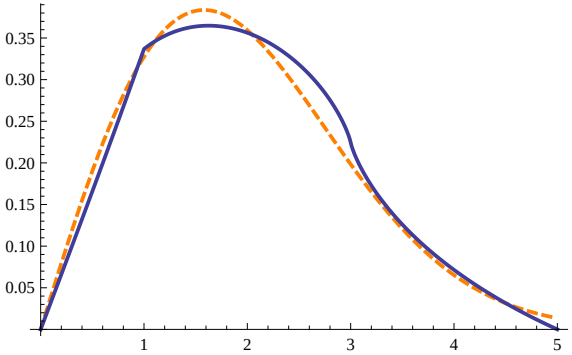

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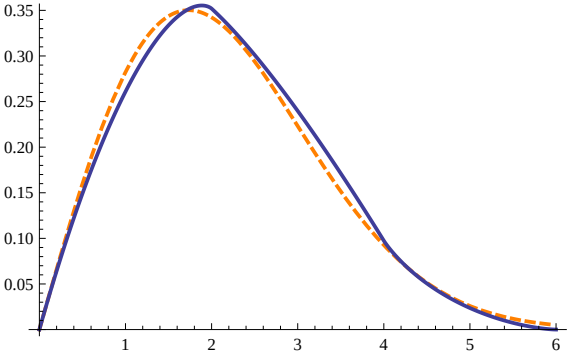

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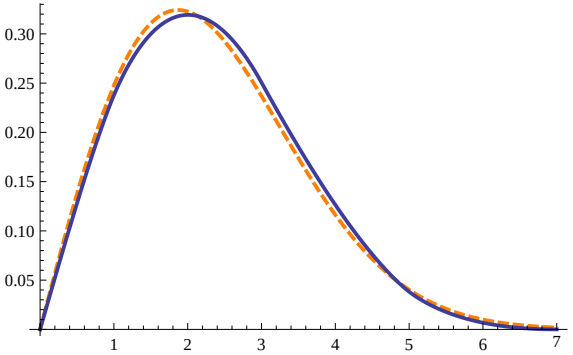

Classical results on the densities

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \operatorname{Re}\left(\frac{\sqrt{x}}{\pi^2}\,K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right)$$ G. J. Bennett 1905

$$p_4(x) = \,??$$

$$\vdots$$

$$p_n(x) = \int_0^\infty xtJ_0(xt)J_0^n(t)\,\mathrm{d}t$$ J. C. Kluyver 1906

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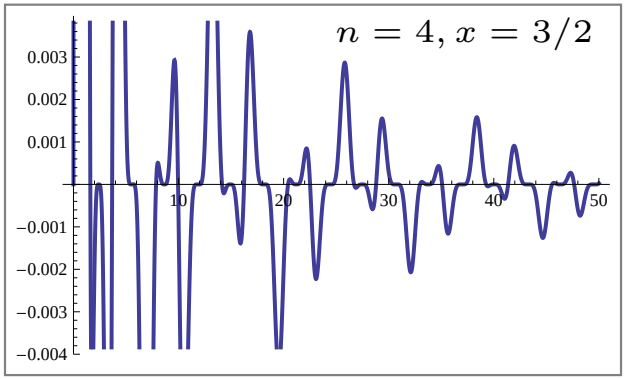

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = ?$$

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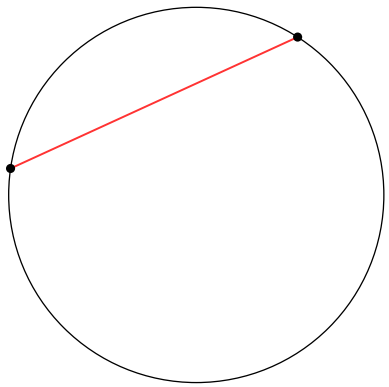
- This is the average length of a random arc on a unit circle.



The moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

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- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.574597237551893657 49 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

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- Hard to evaluate numerically to high precision.
  Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where N is the number of sample points.

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2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
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$W_2(1) = \frac{4}{\pi}$

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The even moments

n	$s=0$	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	OEIS
2	1	2	6	20	70	252	A000984
3	1	3	15	93	639	4653	A002893
4	1	4	28	256	2716	31504	A002895
5	1	5	45	545	7885	127905	A169714
6	1	6	66	996	18306	384156	A169715

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$ **Apéry-like**

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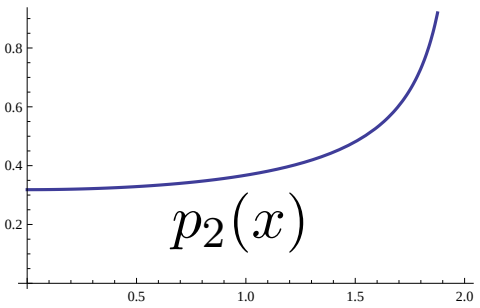
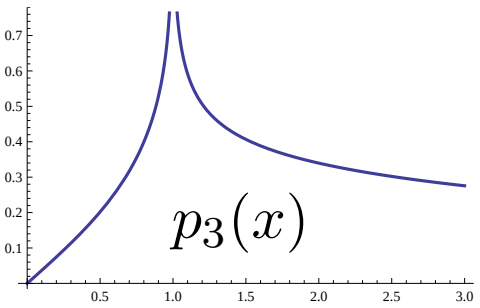
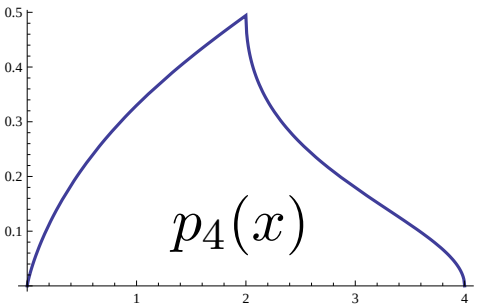
THM Borwein-Nuyens-S-Wan, 2010

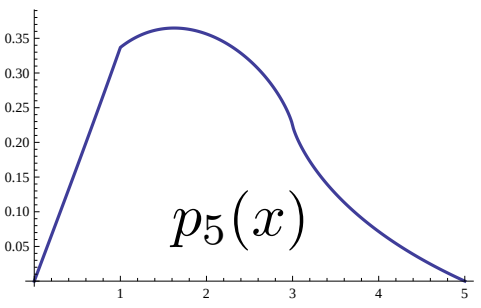
$$W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

Densities of random walks









$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi}\,\frac{x}{(3+x^2)}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\right|\frac{x^2\left(9-x^2\right)^2}{(3+x^2)^3}\right)$$ classical with a spin

$$p_4(x) = \frac{2}{\pi^2}\,\frac{\sqrt{16-x^2}}{x}\,\operatorname{Re}\,{}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\\frac{5}{6},\frac{7}{6}\end{matrix}\right|\frac{\left(16-x^2\right)^3}{108x^4}\right)$$ new BSWZ 2011

$$p_5'(0) = \frac{\sqrt{5}}{40\pi^4}\,\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Mahler measure and random walks

DEF (Logarithmic) Mahler measure of $p(x_1, \ldots, x_n)$:

$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

- $W_n(s) = \displaystyle\int_{[0,1]^n} \left| e^{2\pi i t_1} + \ldots + e^{2\pi i t_n} \right|^s \mathrm{d}\boldsymbol{t}$

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$$W_n'(0) = \mu(x_1 + \ldots + x_n) = \mu(1 + x_1 + \ldots + x_{n-1})$$

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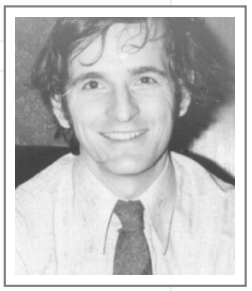

EG Smyth, 1981

$$\mu(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = W_3'(0)$$

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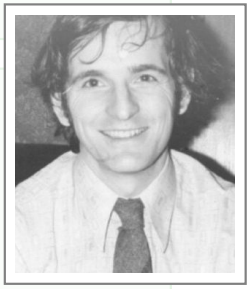


CONJ Rodriguez-Villegas

$$W_5'(0) \stackrel{?}{=} \left(\frac{\sqrt{-15}}{2\pi i}\right)^5 3!\, L(g_{15}, 4) = -L'(g_{15}, -1)$$

$g_{15} = \eta(3\tau)^3\eta(5\tau)^3 + \eta(\tau)^3\eta(15\tau)^3$

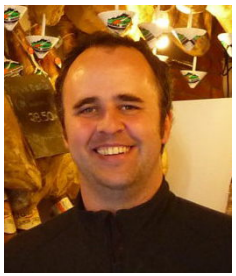
$$W_6'(0) \stackrel{?}{=} 8\left(\frac{\sqrt{-6}}{2\pi i}\right)^6 4!\, L(g_6, 5) = -8L'(g_6, -1)$$

$g_6 = \eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$

Ramanujan-type series for $1/\pi$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots$$

based on joint work with:



Mathew Rogers
(Université de Montréal,
now: data scientist)

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$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

- Starred in High School Musical, a 2006 Disney production

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

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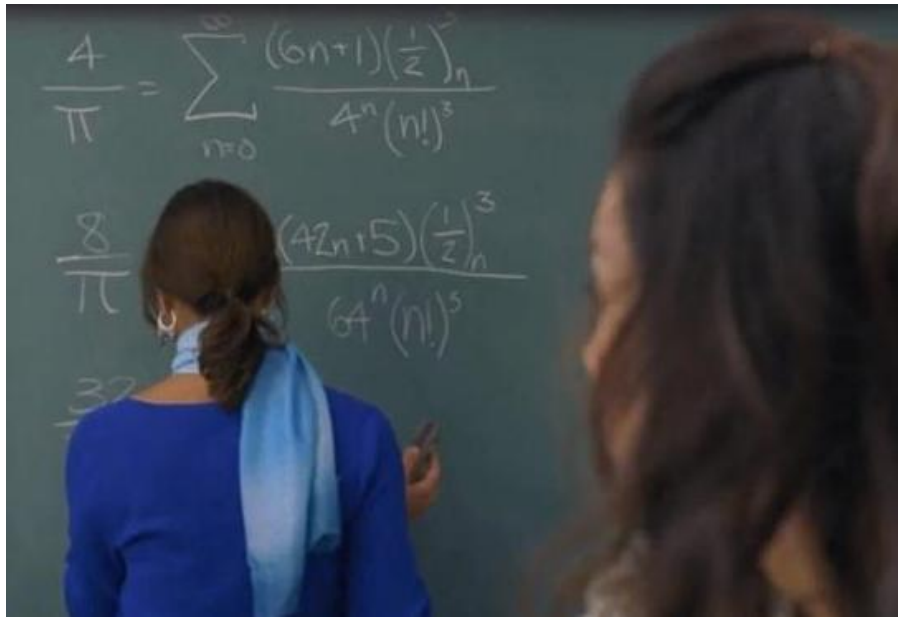

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Another one of Ramanujan's series

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}$$

- Used by R. W. Gosper in 1985 to compute $17,526,100$ digits of π

  Correctness of first 3 million digits showed that the series sums to $1/\pi$ in the first place.

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- First proof of all of Ramanujan's 17 series for $1/\pi$ by Borwein brothers

Jonathan M. Borwein and Peter B. Borwein
Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity
Wiley, 1987

Apéry-like numbers and series for $1/\pi$

- Sato observed that series for $\frac{1}{\pi}$ can be built from Apéry-like numbers:

EG Chan-Chan-Liu 2003

For the Domb numbers $D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$,

$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} D(n) \frac{5n+1}{2^{6n}}.$$

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- Sun offered a \$520 bounty for a proof the following series:

THM Rogers-S 2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n+233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

A brief guide to proving series for $1/\pi$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\substack{\text{modular}\\ \text{function}}} = \underbrace{f(\tau)}_{\substack{\text{modular}\\ \text{form}}} .$$

- Then:

$$\sum_{n=0}^{\infty} a_n (A+Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

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FACT

- For $\tau \in \mathbb{Q}(\sqrt{-d})$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a **quasimodular** form.
- Prototypical $E_2(\tau)$ satisfies $\tau^{-2} E_2\left(-\frac{1}{\tau}\right) - E_2(\tau) = \frac{6}{\pi i \tau}$.

- These are the main ingredients for series for $1/\pi$. Mix and stir.

Positivity of rational functions

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}$$

based on joint work with:

Wadim Zudilin
(Radboud U., NL)

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1,\ldots,n_d \geqslant 0} a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1,\ldots,n_d} > 0$ for all indices.

EG The following rational functions are positive.

$$S(x,y,z) = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

Szegő '33
Kaluza '33
Askey–Gasper '72
S '08

$$A(x,y,z) = \frac{1}{1-(x+y+z)+4xyz}$$

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

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- Both functions are on the boundary of positivity.
- The diagonal coefficients of A are the **Franel numbers** $\sum_{k=0}^{n} \binom{n}{k}^3$.

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CONJ Kauers-Zeilberger 2008 The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

- Would imply conjectured positivity of Lewy–Askey function

$$\frac{1}{(1-x)(1-y)+(1-x)(1-z)+\ldots+(1-z)(1-w)}.$$

Non-negativity proved by a very general result of Scott–Sokal ('14)

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PROP S-Zudilin 2013

The Kauers–Zeilberger function has diagonal coefficients

$$d_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2.$$

Positivity of rational functions

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

EG

- $\dfrac{1}{1-(x+y)}$ is positive.
- $\dfrac{1}{1+x+y}$ has positive diagonal but is not positive.

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THM S-Zudilin 2013

$$F(x,y) = \frac{1}{1+c_1(x+y)+c_2xy}$$ is positive

$\iff$ diagonal of F and $F|_{y=0}$ are positive

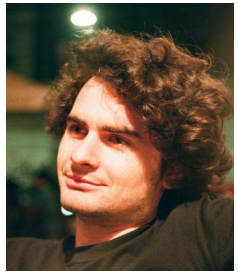
Diagonal and constant term representations

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \operatorname{diag} \frac{1}{(1-x-y)(1-z-w)-xyzw}$$

$$= \operatorname{ct}\left[\left(\frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz}\right)^n\right]$$

based on joint work with:

Alin Bostan
(Université Paris-Saclay)



Sergey Yurkevich
(University of Vienna)

A simple example

EG
constant term

$$\binom{2n}{n} = [x^n]\,(1+x)^{2n}$$

A simple example

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$$\binom{2n}{n} = [x^n]\,(1+x)^{2n} = \operatorname{ct}[\boldsymbol{P}^n], \qquad \boldsymbol{P}(x) = \frac{(1+x)^2}{x}.$$

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EG diagonal

$\binom{2n}{n}$ is the diagonal of $\dfrac{1}{1-x-y}$

$$\sum_{n_1,\ldots,n_d \geqslant 0} a(n_1,\ldots,n_d)\, x_1^{n_1}\cdots x_d^{n_d}$$

multivariate series

$a(n,\ldots,n)$

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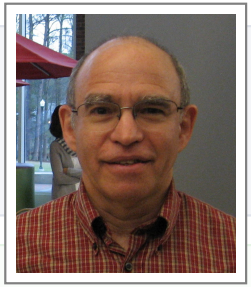
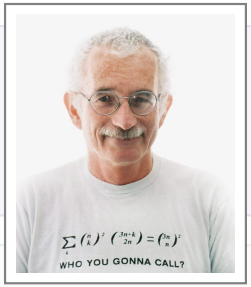
$$\sum_{n_1,\ldots,n_d\geqslant 0} a(n_1,\ldots,n_d)\, x_1^{n_1}\cdots x_d^{n_d} \qquad a(n,\ldots,n)$$

multivariate series — diagonal

THM Gessel, Zeilberger, Lipshitz 1981–88

Diagonals of rational functions are P-recursive.





HW Constant terms are always diagonals.

Ramanujan's elliptic functions

- Berndt, Bhargava & Garvan (1995) develop Ramanujan's theories of elliptic functions based on the hypergeometric functions

$$ {}_2F_1\left(\frac{1}{m}, 1-\frac{1}{m}; 1; x\right), \qquad m \in \{2,3,4,6\}. $$

($m=2$: classical; $m=3,4,6$: alternative bases)



LEM Bostan, S, Yurkevich '23

Let $A_m(n) = \dfrac{\left(\frac{1}{m}\right)_n \left(1-\frac{1}{m}\right)_n}{n!^2}$ where $m \geqslant 2$ is an integer.

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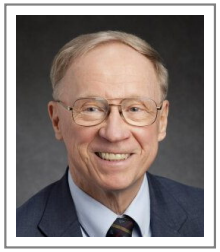

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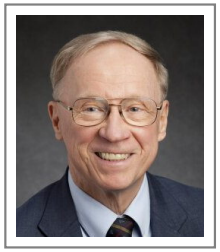

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1. $A_m(n)$ is a **diagonal** for all $m \geqslant 2$.
2. $A_m(n)$ is a **constant term** if and only if $m \in \{2,3,4,6\}$.

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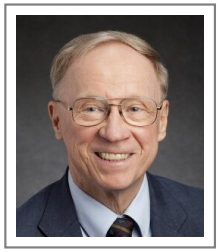

LEM Bostan, S, Yurkevich '23

Let $A_m(n) = \dfrac{\left(\frac{1}{m}\right)_n \left(1-\frac{1}{m}\right)_n}{n!^2}$ where $m \geqslant 2$ is an integer.

1. $A_m(n)$ is a **diagonal** for all $m \geqslant 2$.
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EG $m=3$

$$ 3^{3n} A_3(n) = \frac{(3n)!}{n!^3} = \binom{2n}{n}\binom{3n}{n} = \operatorname{ct}\left[\left(\frac{(1+x)^2(1+y)^3}{xy}\right)^n\right] $$

EG $m=5$

$5^{3n} A_5(n) = 1, 20, 1350, 115500, 10972500, \ldots$ is an integer sequence and diagonal but not a constant term.

Homework

- Such classifications are generally not straightforward!

EG
open!

Is the following hypergeometric sequence a constant term?

$$A(n) = \frac{(8n)!n!}{(4n)!(3n)!(2n)!} = \binom{8n}{4n}\binom{4n}{n}\binom{2n}{n}^{-1}$$

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Characterizations of diagonals

EG Diagonals of rational functions

- $F(x)$ $=$ C-finite sequences

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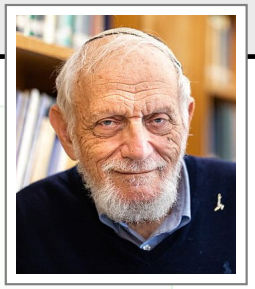

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To see the latter, express the diagonal as $\frac{1}{2\pi i}\int_{|x|=\varepsilon} F\left(x, \frac{z}{x}\right)\frac{\mathrm{d}x}{x}$.

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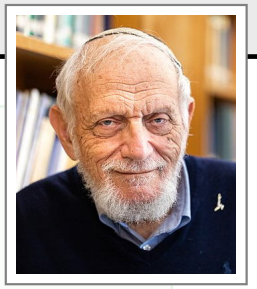

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THM Bostan, Lairez, Salvy '17

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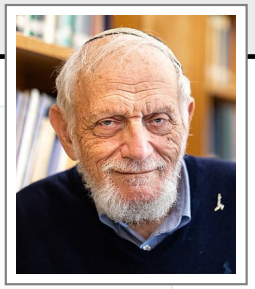

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CONJ Christol '90

Diagonals of rational functions over $\mathbb{Q}$ = globally bounded, P-recursive sequences (⊆ known)

(i.e. $cd^n a_n \in \mathbb{Z}$ for $c,d \in \mathbb{Z}$ and at most exponential growth)

Application: Integrality of P-recursive sequences

- A sequence is P-**recursive** / holonomic if it satisfies a linear recurrence with polynomial coefficients.

EG The **Apéry numbers** $A(n)$ satisfy $A(0) = 1$, $A(1) = 5$ and

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

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- OGFs of such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically leads to **congruences** such as

$$A(n) \equiv \begin{cases} 1 \pmod 8, & \text{if } n \text{ even}, \\ 5 \pmod 8, & \text{if } n \text{ odd}. \end{cases}$$

Chowla–Cowles–Cowles '80
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$$\sum_{n \geqslant 0} A(n) t^n = \frac{17-t-z}{4\sqrt{2}(1+t+z)^{3/2}} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -\frac{1024t}{(1-t+z)^4} \right), \quad z = \sqrt{1-34t+t^2}.$$

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EG constant term

$A(n) = \operatorname{ct}[L^n]$ with $L = \dfrac{(1+y)(1+z)(1+x+z)(1+x+z+yz)}{xyz}$

- $F_A(t) = \displaystyle\sum_{n \geqslant 0} A(n) t^n = \operatorname{ct}\left[\frac{1}{1-tL}\right]$ is a **period function**.

  The DE satisfied by $F_A(t)$ is the **Picard–Fuchs DE** for the family $V_t : 1 - tL = 0$.

  Generically, V_t is birationally equivalent to a **K3 surface** with Picard number 19. (Beukers–Peters '84)

A question of Zagier

- $c(n)$ is a **constant term** if $c(n) = \operatorname{ct}[P^n(\boldsymbol{x})Q(\boldsymbol{x})]$ Rowland–Zeilberger '14
  for Laurent polynomials $P, Q \in \mathbb{Q}[\boldsymbol{x}^{\pm 1}]$ in $\boldsymbol{x} = (x_1, \ldots, x_d)$.

EG $Q = 1$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \operatorname{ct}\left[\left(\frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz}\right)^n\right]$$

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$$\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1} = \operatorname{ct}\left[\left(\frac{(x+1)^2}{x}\right)^n (1-x)\right]$$

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Which are linear combinations of constant terms?

- We will answer this in the case of a single variable.
- For instance: Are Fibonacci numbers constant terms? (C-finite sequences!) $\dfrac{x}{1 - x - x^2}$

C-finite sequences that are constant terms

- C-finite sequences:

$$\underset{\text{(finite support)}}{A_0(n)} + \sum_{j=1}^{d} \sum_{r=0}^{m_j-1} c_{j,r} n^r \lambda_j^n \qquad \text{(characteristic roots } \lambda_j\text{)}$$

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  And so are sequences of finite support ($\lambda = 0$).

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- $2^n = \mathrm{ct}\,[(x+2)^n] = \mathrm{ct}\,[2^n]$
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EG Fibonacci numbers are not (sums of) constant terms.

EG $2^n + 1$ is not a constant term but is a sum of two.

Example: Fibonacci numbers

- Our key ingredient to answer these questions are **congruences**:

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If $A(n)$ is a constant term then, for all large enough primes p,

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It follows that

$$F(p) \equiv \begin{cases} 1, & \text{if } p \equiv 1, 4 \bmod 5, \\ -1, & \text{if } p \equiv 2, 3 \bmod 5, \end{cases} \pmod{p}.$$

Hence, the Fibonacci numbers cannot be constant terms.

Hypergeometric sequences

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LEM Bostan, S, Yurkevich '23

This hypergeometric sequence is not a constant term (or a linear combination of constant terms).

Proof idea: $A(p)$ takes different values modulo p depending on whether $p \equiv \pm 1 \pmod 9$.

Constant terms are special

- For hypergeometric sequences: (or C-finite or P-recursive)

$$\{\underset{\text{(or linear combinations)}}{\text{constant terms}}\} \subsetneq \{\text{diagonals}\} \subseteq \{P\text{-recursive, globally bounded seq's}\}$$

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- The following is an indication that constant terms are special among diagonals and often have significant additional arithmetic properties.

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2. $A_m(n)$ is a constant term if and only if $m \in \{2,3,4,6\}$.

- The cases $m \in \{2,3,4,6\}$ correspond to the hypergeometric functions underlying Ramanujan's theory of elliptic functions.
($m=2$: classical case; $m=3,4,6$: alternative bases)

Collecting some thoughts...

- Constant terms are an arithmetically interesting subset of diagonals.
- We have classified them in the case of a single variable. Natural classes of sequences to consider next:
  - Hypergeometric sequences
  - Algebraic sequences (diagonals in two variables)
  - Algebraic hypergeometric series
  - Integral factorial ratios

(Bober, 2007; via Beukers–Heckman)

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Is $A(n) = \dfrac{(8n)!n!}{(4n)!(3n)!(2n)!} = \dbinom{8n}{4n}\dbinom{4n}{n}\dbinom{2n}{n}^{-1}$ a constant term?

$$1, 140, 60060, 29745716, 15628090140, \ldots = \operatorname{ct}\left[\left(\frac{(1+x)^8}{(1-x)^2x^3}\right)^n\right]$$

This is algebraic (and therefore a diagonal) and hypergeometric.

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- We have classified them in the case of a single variable. Natural classes of sequences to consider next:
  - Hypergeometric sequences
  - Algebraic sequences (diagonals in two variables)
  - Algebraic hypergeometric series
  - Integral factorial ratios

(Bober, 2007; via Beukers–Heckman)

EG

Is $A(n) = \dfrac{(8n)!n!}{(4n)!(3n)!(2n)!} = \binom{8n}{4n}\binom{4n}{n}\binom{2n}{n}^{-1}$ a constant term?

$$1, 140, 60060, 29745716, 15628090140, \ldots = \operatorname{ct}\left[\left(\frac{(1+x)^8}{(1-x)^2x^3}\right)^n\right]$$

This is algebraic (and therefore a diagonal) and hypergeometric.

- How to find representations as (nice) constant terms or diagonals?

  Once found, such representations can be proved using **creative telescoping**.
- How unique are the Laurent polynomials in a constant term?

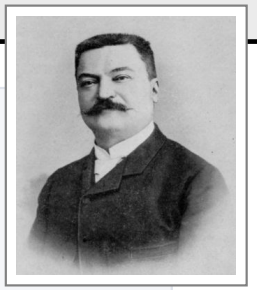
  Connections to cluster algebras, mutations of Laurent polynomials, ...

Gessel–Lucas congruences

$$A(pn+k) \equiv A(k)A(n) + pnA'(k)A(n) \pmod{p^2}$$

A. Straub
Gessel-Lucas congruences for sporadic sequences
Monatshefte für Mathematik, Vol. 203, 2024, p. 883–898

Lucas congruences



THM
Lucas
1878

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots \pmod{p},$$

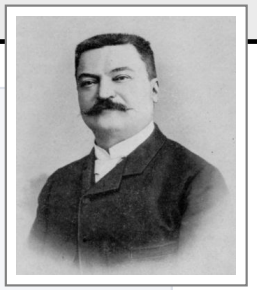
where n_i and k_i are the p-adic digits of n and k.

EG

$$\binom{136}{79} \equiv \binom{3}{2}\binom{5}{4}\binom{2}{1} = 3\cdot 5\cdot 2 \equiv 2 \pmod{7}$$

LHS $= 100922074694299394627152562728591193280 0$

Lucas congruences



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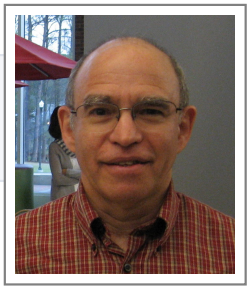
- Interesting sequences like the **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy such **Lucas congruences** as well:



THM Gessel '82

$$A(n) \equiv A(n_0)A(n_1)\cdots A(n_r) \pmod{p}$$

Application: Primes not dividing Apéry numbers

CONJ Rowland–Yassawi '15

There are infinitely many primes p such that p does not divide any Apéry number $A(n)$.
Such as $p = 2, 3, 7, 13, 23, 29, 43, 47, \ldots$

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- Heuristically, combine Lucas congruences,
- palindromic behavior of Apéry numbers, that is

$$A(n) \equiv A(p-1-n) \pmod{p},$$

- and $e^{-1/2} = \lim_{p\to\infty} \left(1 - \frac{1}{p}\right)^{(p+1)/2}$.

Gessel–Lucas congruences

- Lucas congruences: $A(pn+k) \equiv A(n)A(k) \pmod{p}$

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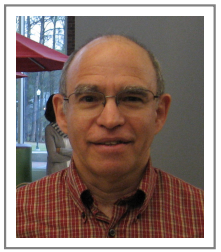

- In the case of the Apéry numbers, Gessel ('82) observed that these congruences can be extended modulo p^2.

THM S '24 All of the $6+6+3$ known sporadic sequences satisfy **Gessel–Lucas congruences** modulo every odd prime:

$$A(pn+k) \equiv A(k)A(n) + pnA'(k)A(n) \pmod{p^2}$$

- Here, $A'(n)$ is the formal derivative of $A(n)$.
  These are rational numbers!

The formal derivative of recurrence sequences

- Suppose $A(n)$ is the unique solution for all $n \geqslant 0$ to

$$\sum_{j=0}^{r} c_j(n) A(n-j) = 0 \qquad \text{with } A(0) = 1 \text{ and } A(j) = 0 \text{ for } j < 0.$$

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Note Let $F(x) = \sum_{n \geqslant 0} A(n) x^n$ and $G(x) = \sum_{n \geqslant 1} A'(n) x^n$.

Then the corresponding differential equation satisfied by $F(x)$ is also solved by $\log(x) F(x) + G(x)$.

The formal derivative of recurrence sequences: example

- $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ is the unique solution with $A(0) = 1$ to:

$$(n+1)^2 A(n+1) = (11n^2 + 11n + 3)A(n) + n^2 A(n-1)$$

- Then $A'(n)$ is the unique solution with $A'(0) = 0$ to:

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EG $A'(1), A'(2), \ldots = 5, \frac{75}{2}, \frac{1855}{6}, \frac{10875}{4}, \frac{299387}{12}, \frac{943397}{4}, \frac{63801107}{28}, \ldots$

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- Since the interpolation satisfies the continuous version of the recurrence:

$$A'(n) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k} \Bigg|_{x=n}$$
$$= 5 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} (H_n - H_k)$$

Approaches to proving Lucas congruences

- From suitable expressions as a **binomial sum**. Gessel '82, McIntosh '92

Apéry numbers: $\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$

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THM Samol, van Straten '09

Suppose the origin is the only interior integral point of the Newton polytope of $P \in \mathbb{Z}[\boldsymbol{x}^{\pm 1}]$.

Then $A(n) = \operatorname{ct}[P(\boldsymbol{x})^n]$ satisfies Lucas congruences.

$$P = \frac{(x+y)(z+1)(x+y+z)(y+z+1)}{xyz}$$

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For instance, diagonals of $1/Q(\boldsymbol{x})$ for $Q(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$ with $Q(\boldsymbol{x})$ linear in each variable and $Q(\mathbf{0}) = 1$.

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- From suitable **modular parametrizations**. Beukers–Tsai–Ye '25

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THM Gorodetsky '21

Each sporadic sequence, except possibly (η), can be expressed as $\operatorname{ct}[P(\boldsymbol{x})^n]$ so that the result of Samol–van Straten applies.

EG Gorodetsky '21

$$(\eta)\colon \frac{(zx+xy-yz-x-1)(xy+yz-zx-y-1)(yz+zx-xy-z-1)}{xyz}$$

$(1,0,0)$, $(1,1,0)$ and their permutations are interior points.

Q Algorithmic tools to find (useful) constant term expressions?

Once found, algorithmically provable using creative telescoping.

Lucas congruences in terms of the GF

- Given $F(x) = \sum_{n=0}^{\infty} A(n)x^n$, we write $F_p(x) = \sum_{n=0}^{p-1} A(n)x^n$ for its p-**truncation**.

LEM $A(n)$ satisfies Lucas congruences modulo p
$\iff$ $\frac{1}{F^{p-1}(x)}$ modulo p is a polynomial of degree $< p$.

proof

$$A(n) \equiv A(n_0)A(n_1)A(n_2)\cdots \pmod{p}$$

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Since the first p coefficients of $\ldots$ always match, the final congruence is equivalent to the RHS being a polynomial of degree $\leqslant p-1$. □

Lucas congruences via modular forms

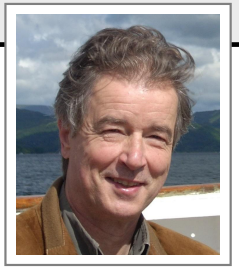
- Suppose $F(x) = \sum_{n=0}^{\infty} A(n) x^n$ has **modular parametrization**:
  $F(x)$ is a modular form for some modular function $x(\tau)$.

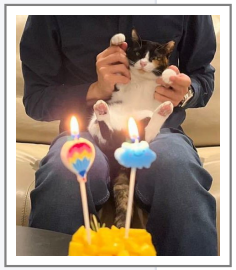
THM Beukers–Tsai–Ye '25

Suppose that:

- $x(\tau) = q + q^2\mathbb{Z}[[q]]$ with $q = \mathrm{e}^{2\pi i\tau}$ is a **Hauptmodul** for $\Gamma = \Gamma_0(N)$ (or Atkin–Lehner extension).
- $F(x(\tau)) = 1 + q\mathbb{Z}[[q]]$ is a weight 2 modular form for Γ.
- $F(x(\tau))$ has a unique zero at $[\tau_0]$ of order $\leqslant 1$, where $[\tau_0]$ is the (unique) pole of $x(\tau)$.

Then $A(n)$ satisfies the Lucas congruences for all primes p.





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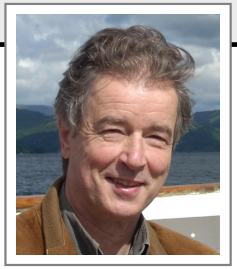

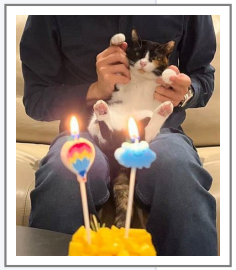

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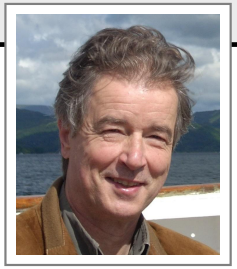
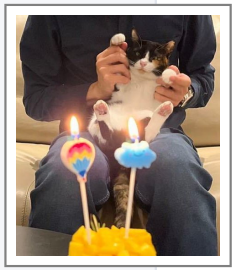
proof Suppose $E(\tau)$ is a modular form for Γ with weight $2(p-1)$ such that $E(\tau) \equiv 1 \pmod{p}$.

Lucas congruences via modular forms





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  $F(x)$ is a modular form for some modular function $x(\tau)$.

THM Beukers–Tsai–Ye '25

Suppose that:

- $x(\tau) = q + q^2\mathbb{Z}[[q]]$ with $q = \mathrm{e}^{2\pi i\tau}$ is a **Hauptmodul** for $\Gamma = \Gamma_0(N)$ (or Atkin–Lehner extension).
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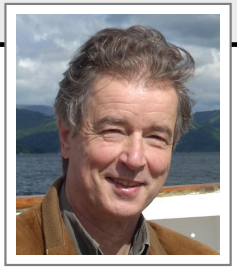
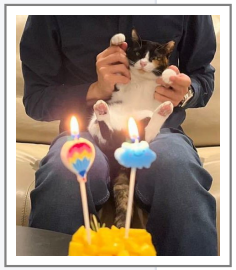
Lucas congruences via modular forms





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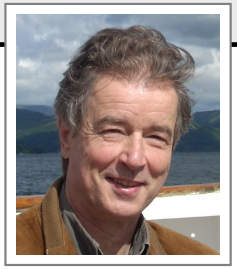

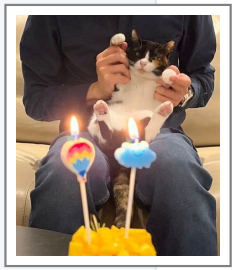

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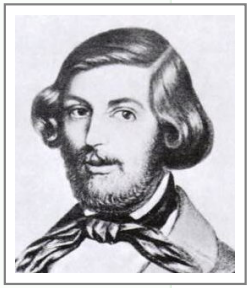
Lucas congruences via modular forms, cont'd

- Needed: weight $2(p-1)$ modular form $E(\tau)$ for Γ with $E(\tau) \equiv 1 \pmod{p}$.

EG The normalized **Eisenstein series**

$$E_k(\tau) = 1 + \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1-q^n}$$



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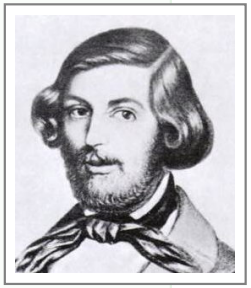

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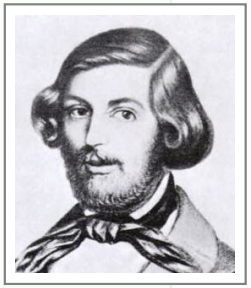

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- If $p \geqslant 5$ and Γ is $\Gamma_0(N)$ extended by $\tau \to -\frac{1}{N\tau}$:

  $$E(\tau) := \tfrac{1}{2} \left[E_{p-1}(\tau)^2 + N^{p-1} E_{p-1}(N\tau)^2 \right]$$

Sporadic sequences mod p^r are automatic

THM Rowland, Yassawi '15

If an integer sequence $A(n)$ is the diagonal of $F(\boldsymbol{x}) \in \mathbb{Z}(\boldsymbol{x})$, then the reductions $A(n) \pmod{p^r}$ are p-**automatic**.

Constructive proof of results by Denef and Lipshitz '87.

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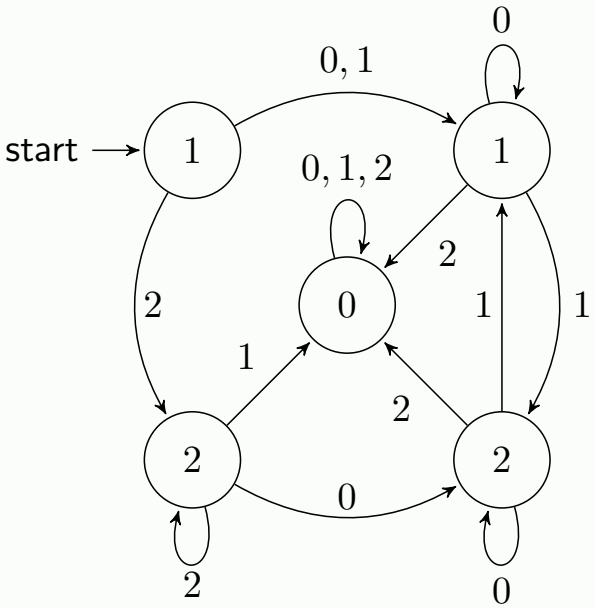

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Instead via automaton:

$35 = 1\ 0\ 2\ 2$ in base 3

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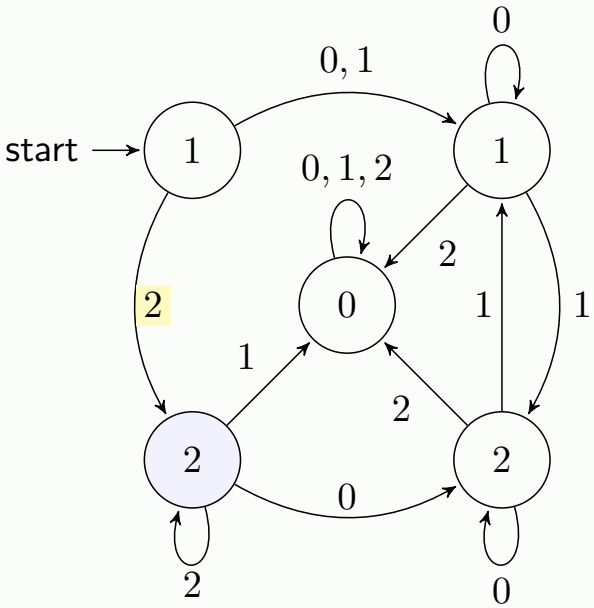

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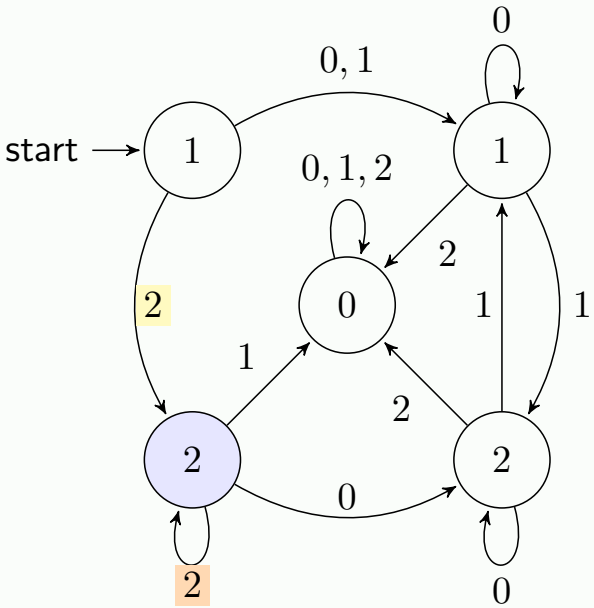

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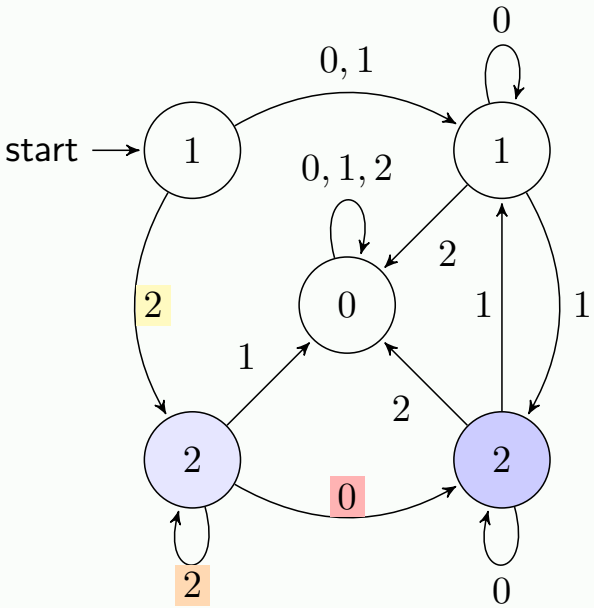

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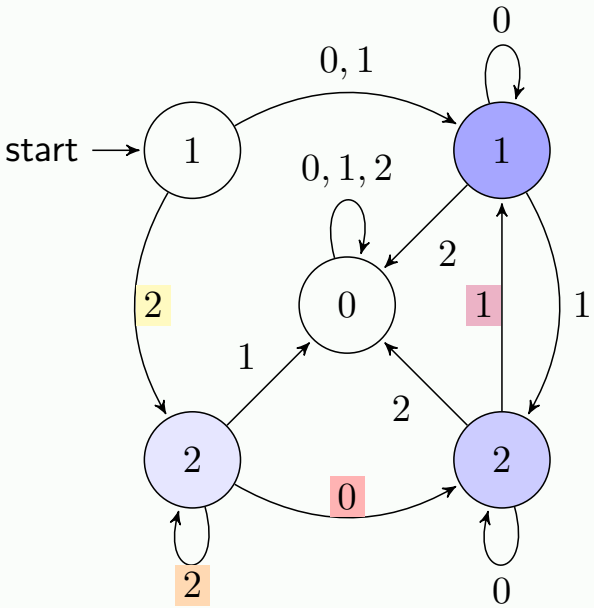

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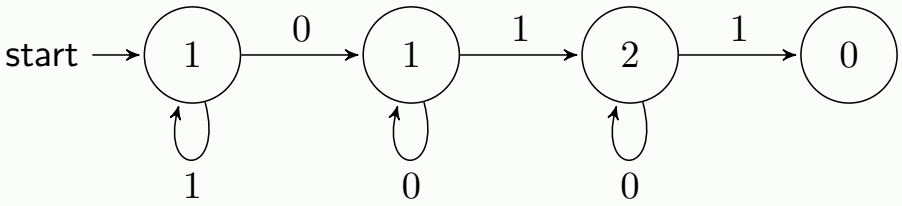

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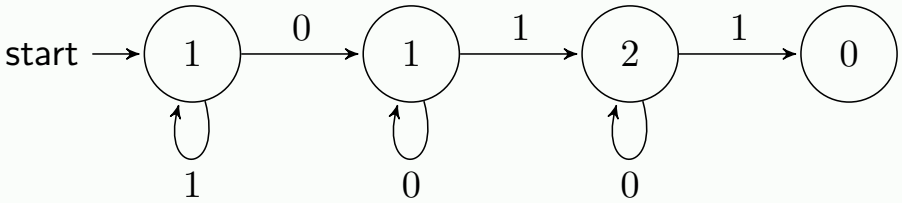
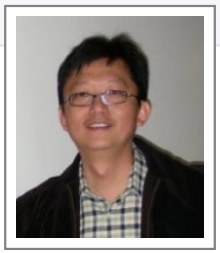


THM Eu, Liu, Yeh '08

$$C(n) \equiv \begin{cases} 1, & \text{if } n = 2^a - 1 \text{ for some } a \geqslant 0, \\ 2, & \text{if } n = 2^b + 2^a - 1 \text{ for some } b > a \geqslant 0, \\ 0, & \text{otherwise,} \end{cases} \pmod 4.$$



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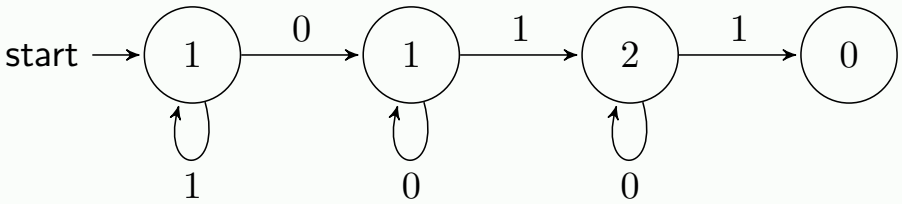

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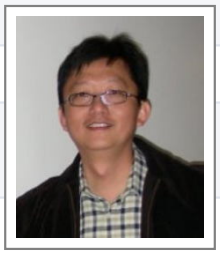


COR $C(n) \not\equiv 3 \pmod 4$

Things quickly get more complicated

- Liu–Yeh (2010) also determine the Catalan numbers modulo 16 and 64.

Theorem 5.5. *Let c_n be the n-th Catalan number. First of all, $c_n \not\equiv_{16} 3, 7, 9, 11, 15$ for any n. As for the other congruences, we have*

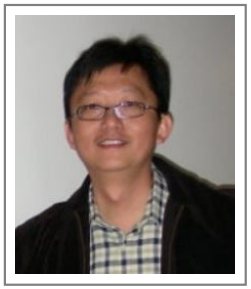
$$c_n \equiv_{16} \begin{cases} \left.\begin{matrix}1\\5\\13\end{matrix}\right\} & \text{if } d(\alpha)=0 \text{ and } \begin{cases}\beta \le 1,\\ \beta = 2,\\ \beta \ge 3,\end{cases} \\ \left.\begin{matrix}2\\10\end{matrix}\right\} & \text{if } d(\alpha)=1,\ \alpha=1 \text{ and } \begin{cases}\beta = 0 \text{ or } \beta \ge 2,\\ \beta = 1,\end{cases} \\ \left.\begin{matrix}6\\14\end{matrix}\right\} & \text{if } d(\alpha)=1,\ \alpha\ge 2 \text{ and } \begin{cases}(\alpha=2, \beta\ge 2) \text{ or } (\alpha\ge 3, \beta\le 1),\\ (\alpha=2, \beta\le 1) \text{ or } (\alpha\ge 3, \beta\ge 2),\end{cases} \\ \left.\begin{matrix}4\\12\end{matrix}\right\} & \text{if } d(\alpha)=2 \text{ and } \begin{cases}zr(\alpha)\equiv_2 0,\\ zr(\alpha)=1,\end{cases} \\ 8 & \text{if } d(\alpha)=3, \\ 0 & \text{if } d(\alpha)\ge 4. \end{cases}$$

where $\alpha = (CF_2(n+1)-1)/2$ and $\beta = \omega_2(n+1)$ (or $\beta = \min\{i \mid n_i = 0\}$).

$\omega_p(n) = p$-adic valuation of n

$CF_p(n) = n \,/\, p^{\omega_p(n)}$

$d(n) =$ sum of 2-adic digits of n



- For comparison: the corresponding minimal automaton has 26 states.

Catalan numbers: forbidden residues

EG Rowland, Yassawi '15

$$C(n) \not\equiv 3 \pmod 4$$ Eu–Liu–Yeh '08

$$C(n) \not\equiv 9 \pmod{16}$$ Liu–Yeh '10

$$C(n) \not\equiv 17, 21, 26 \pmod{32}$$

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Let $P(r)$ be the proportion of residues not attained by $C(n)$ mod 2^r.
Does $P(r) \to 1$ as $r \to \infty$?

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r	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$P(r)$	0	.25	.25	.31	.41	.47	.54	.59	.65	.69	.73	.76	.79	.82
$N(r)$	0	1	2	5	13	30	69	152	332	710	1502	3133	6502	13394
$A(r)$	0	1	0	1	3	4	9	14	28	46	82	129	236	390

$N(r) = \#$ residues not attained mod 2^r

$A(r) = \#$ additional residues not attained mod $2^r = N(r) - 2N(r-1)$

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CONJ Bostan '15

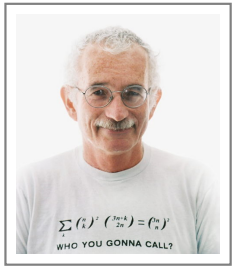
$C(n) \not\equiv 3 \pmod{10}$ for all $n \geqslant 0$.

$C(n) \not\equiv 1, 7, 9 \pmod{10}$ for sufficiently large n.

If true, the last digit of any sufficiently large odd Catalan number is always 5. ($n > 255$?)

Linear congruence schemes

- The Catalan numbers $C(n)$ modulo 3 can be described:
  - by an automaton with 4 states (plus a zero state)
  - by a **linear 3-scheme** with 2 states
    (Rowland–Zeilberger '14)



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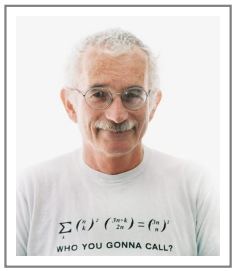
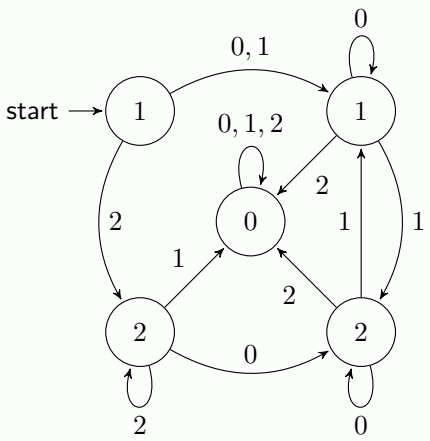


EG
mod 3

automatic 3-scheme



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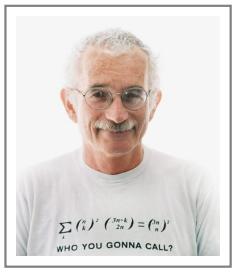

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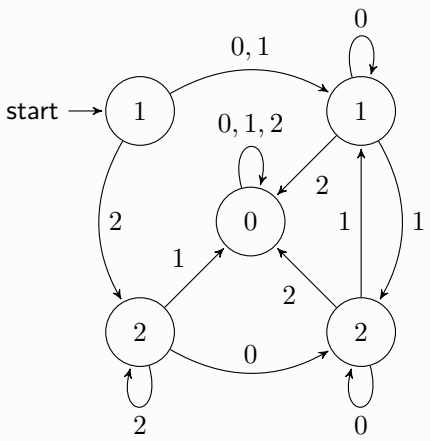

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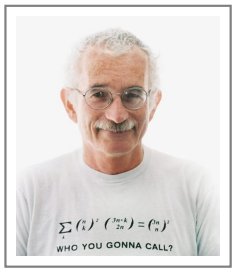

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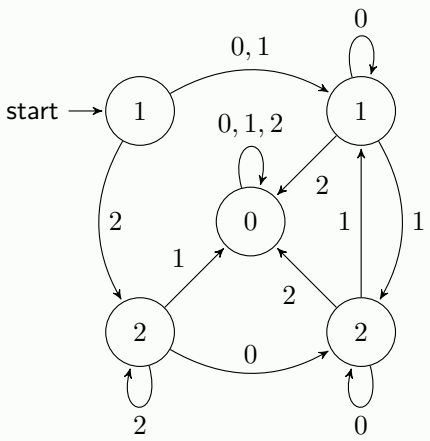

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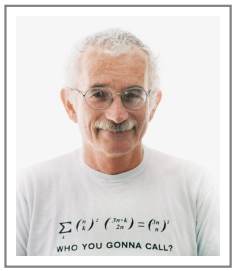

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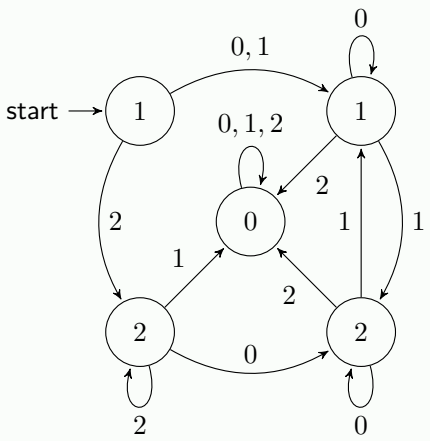

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Lucas congruences:

$$A(pn+k) \equiv A(k)A(n) \pmod{p}$$

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$A(n) \pmod{p}$ satisfies a single-state linear p-scheme (and $A(0)=1$).
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Note Gessel–Lucas congruences are much more rare!
For instance, for $k=0$, we get the **supercongruences**

$$A(pn) \equiv A(n) \pmod{p^2}.$$

Supercongruences for Apéry numbers

- Chowla, Cowles and Cowles (1980) conjectured that, for $p \geqslant 5$,

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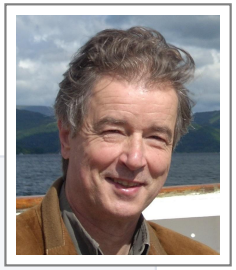

THM Beukers, Coster '85, '88

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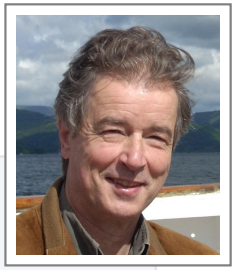

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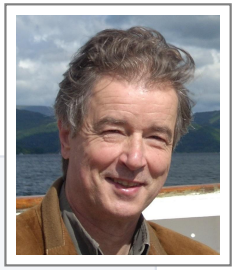

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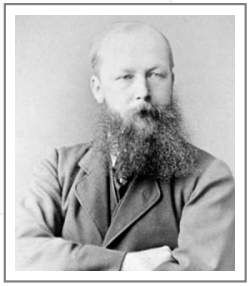
For $p \geqslant 5$, Wolstenholme (1862) showed that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$



Supercongruences for Apéry-like numbers

CONJ Osburn–Sahu '09 All known Apéry-like numbers satisfy supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$


- This is finally proven in all cases.
  For instance, for the six sporadic sequences related to $\zeta(3)$:

$A(n)$	
$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '85-'88
$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '16
$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Amdeberhan–Tauraso '16 ($r=1$) **Alinquant–Osburn '25**
$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	Osburn–Sahu–S '16
$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	Gorodetsky '18

Multivariate supercongruences

- The Almkvist–Zudilin numbers are the sporadic sequence

$$Z(n) = \sum_k (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}.$$

EG S 2014

The Almkvist–Zudilin numbers are the diagonal Taylor coefficients of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}{}^4} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

CONJ S 2014

For $p \geqslant 5$, we have the multivariate supercongruences

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THM Beukers, Houben, S 2018

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q linear in each variable.

The above **Gauss congruences** modulo p^r are satisfied by the coefficients of P/Q if and only if $N(P) \subseteq N(Q)$.

Collecting some thoughts. . .

THM S '24 The known sporadic sequences satisfy the **Gessel–Lucas congruences**

$$A(pn+k) \equiv A(k)A(n) + pnA'(k)A(n) \pmod{p^2}.$$

- Lucas congruences correspond to single-state linear p-schemes.
  Gessel–Lucas congruences are instances of 2-state linear p-schemes.
- It would be of interest to study **few-state p-schemes** systematically:
  - What kind of "generalized Lucas congruences" does one get?
  - Which sequences satisfy such congruences? (mod p, mod p^2?)

  Partial results by Henningsen–S ('22) for certain constant term sequences.
- Are there interesting q-**analogs**?
  - q-Lucas congruences have been studied. Olive '65, Désarménien '82
  - For $k=0$, we get $A(pn) \equiv A(n) \pmod{p^2}$. **(Supercongruences!)**
    q-analogs known for some sporadic sequences. S '19, Gorodetsky '19

Apéry limits and Franel's suspicions

based on joint work(s) with:

Marc Chamberland
(Grinnell College)

Wadim Zudilin
(Radboud U., NL)

M. Chamberland, A. Straub
Apéry limits: Experiments and proofs
American Mathematical Monthly, Vol. 128, Nr. 9, 2021, p. 811-824

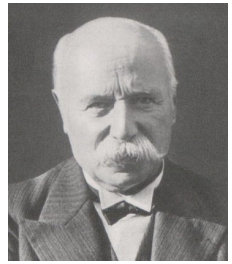
A. Straub, W. Zudilin
Sums of powers of binomials, their Apéry limits, and Franel's suspicions
International Mathematics Research Notices, Vol. 2023, Nr. 11, 2023, p. 9861-9879

Mit der Tür ins Haus fallen. . .

Falling into the house with the door. . .

CONJ Franel, 1895

The minimal recurrence for $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ has order $\lfloor \frac{s+1}{2} \rfloor$.



THM Stoll '97

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OPEN Is that recurrence of minimal order?

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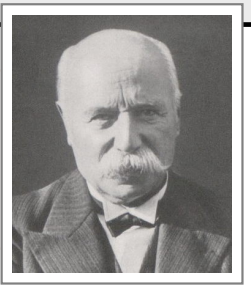

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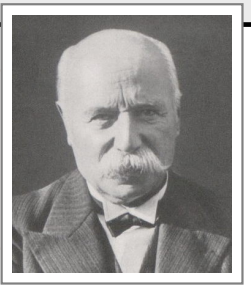

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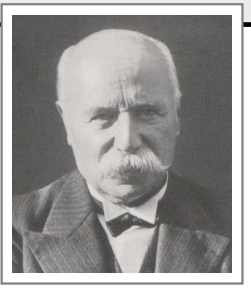

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Hence, $A_j^{(s)}(n)$ with $0 \leqslant 2j < s$ are linearly independent, so that any telescoping recurrence has order at least $\lfloor \frac{s+1}{2} \rfloor$.

Background: Creative telescoping

Goal A **telescoping recurrence** for $\underbrace{\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2}_{=: a(n,k)}$

N, K shift operators in n and k: $Na(n,k) = a(n+1,k)$

Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger
$A = B$
A. K. Peters, Ltd., 1st edition, 1996

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EG

$$P(n,N) = (n+2)^3 N^2 - (2n+3)(17n^2+51n+39)N + (n+1)^3$$

$$R(n,k) = \frac{4k^4(2n+3)(4n^2-2k^2+12n+3k+8)}{(n-k+1)^2(n-k+2)^2}$$

$R(n,k)$ is the **certificate** of the **telescoping recurrence** operator $P(n,N)$.

Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger
$A = B$
A. K. Peters, Ltd., 1st edition, 1996

Background: Poincaré and Perron

- Normalized general homogeneous linear recurrence of order d:

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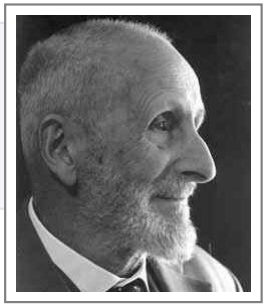
THM Perron 1909

Suppose, in addition, $p_0(n) \neq 0$ for all $n \geqslant 0$.
Then, for each λ_k, there exists a u_n such that (P) holds.



Another look at Apéry's recurrence and limit

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? Tools to construct the solutions guaranteed by Perron's theorem?

Approaches to proving Apéry limits

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1. Via explicit expressions: (Apéry, '78)

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5. Via continued fractions (for recurrences of order 2)

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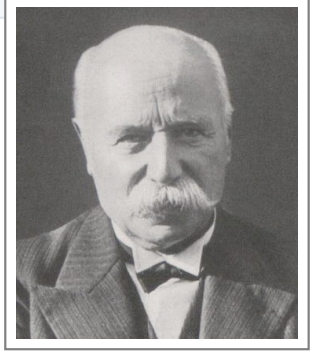
DEF
Franel
1894

$A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ are the (generalized) Franel numbers.



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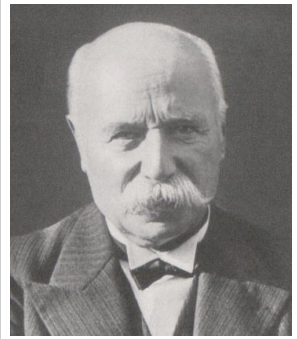

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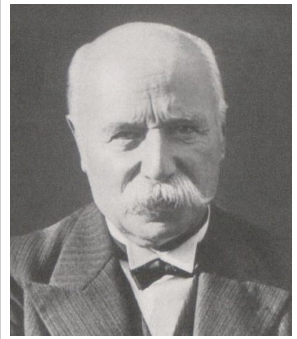

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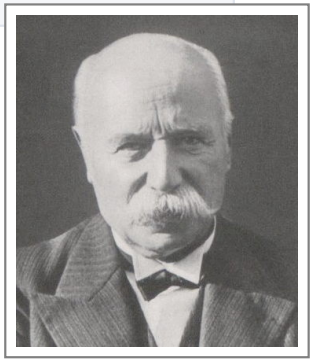

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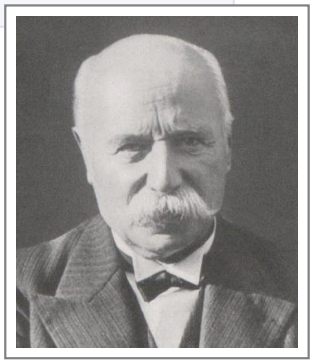

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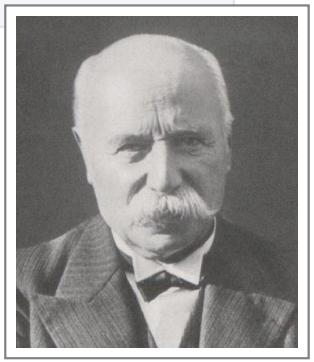

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(spoiler: the degree part is not true)

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- Goal: The minimal **telescoping** recurrence for $A^{(s)}(n)$ has order $\geqslant \lfloor \frac{s+1}{2} \rfloor$.

How to prove lower bounds for orders of recurrences?

EG

- $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$: recurrence of order 2 (Apéry '78)
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  an algorithm to compute order 1 (right) factors of recurrence operators.
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  For Apéry numbers: $\mu = (1+\sqrt{2})^4$.
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Solutions to the Franel number recurrences

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.
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3. $P(n,N)\sum_{k=0}^{n}\binom{n}{k-t}^s = O(t^s)$ omitted terms are $O(t^s)$
4. $A^{(s)}(n,t) = \left(\frac{\pi t}{\sin(\pi t)}\right)^s \sum_{k=0}^{n}\binom{n}{k-t}^s$ and so $P(n,N)A^{(s)}(n,t) = O(t^s)$. □

Apéry limits and lower bounds

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"poof" For large n and $k \approx n/2$,

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- This implies Franel's conjecture on the exact order if the minimal-order recurrence is telescoping. True at least for $s \leqslant 30$.

Collecting some thoughts...

- Applications of Apéry limits:
  - Irrationality proofs for $\zeta(2)$ and $\zeta(3)$
  - Explicitly construct the solutions guaranteed by Perron's theorem
  - Continued fractions
  - Prove lower bounds on orders of recurrences **new!**

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- Many open questions! For instance:
  - Cusick '89 and Stoll '97 construct recurrences for Franel numbers. Can these constructions produce telescoping recurrences?
  - What can we learn from other families of binomial sums?

    Also, it would be nice to simplify some of the technical steps in the arguments.
  - Can we (uniformly) establish the conjectural Apéry limits for CY DE's?
  - Can we explain when CT falls short? And algorithmically "fix" this issue?

Interpolated sequences and critical L-values of modular forms

based on joint work with:

Robert Osburn
(University College Dublin)

R. Osburn, A. Straub
Interpolated sequences and critical L-values of modular forms
Chapter 14 of the book: *Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory*
Editors: J. Blümlein, P. Paule and C. Schneider; Springer, 2019, p. 327-349

L-value interpolations

THM
Ahlgren–Ono 2000

For primes $p > 2$, the Apéry numbers for $\zeta(3)$ satisfy

$$A(\tfrac{p-1}{2}) \equiv a_f(p) \qquad (\mathrm{mod}\, p^2),$$

with $f(\tau) = \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\geqslant 1} a_f(n)q^n \in S_4(\Gamma_0(8))$.

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THM Zagier 2016

$$A(-\tfrac{1}{2}) = \tfrac{16}{\pi^2} L(f,2)$$

- Here, $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$ is absolutely convergent for $x \in \mathbb{C}$.
- Predicted by Golyshev based on motivic considerations, the connection of the Apéry numbers with the double covering of a family of K3 surfaces, and the Tate conjecture.

D. Zagier
Arithmetic and topology of differential equations
Proceedings of the 2016 ECM, 2017

L-value interpolations, cont'd

- Zagier found 6 sporadic integer solutions $C_*(n)$ to: $*$ one of $\boldsymbol{A}$-$\boldsymbol{F}$

$$(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1} \qquad u_{-1}=0, u_0=1$$

THM 1985 - 2019

There exists a weight 3 newform $f_*(\tau) = \sum_{n\geqslant 1} \gamma_{n,*} q^n$, so that

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THM Osburn S '19

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For sequence $\boldsymbol{E}$, $\displaystyle\operatorname*{res}_{x=-1/2} C_{\boldsymbol{E}}(x) = \frac{6}{\pi^2} L(f_{\boldsymbol{E}}, 1).$

L-value interpolations, cont'd

$$C_*(-\tfrac{1}{2}) = \frac{\alpha_*}{\pi^2} L(f_*, 2)$$

*	$C_*(n)$	$f_*(\tau)$	N_*	CM	α_*
A	$\sum_{k=0}^{n} \binom{n}{k}^3$	$\frac{\eta(4\tau)^5\eta(8\tau)^5}{\eta(2\tau)^2\eta(16\tau)^2}$	32	$\mathbb{Q}(\sqrt{-2})$	8
B	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	8
C	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	$\eta(2\tau)^3\eta(6\tau)^3$	12	$\mathbb{Q}(\sqrt{-3})$	12
D	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	16
E	$\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}$	$\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$	8	$\mathbb{Q}(\sqrt{-2})$	6
F	$\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$	$q - 2q^2 + 3q^3 + \ldots$	24	$\mathbb{Q}(\sqrt{-6})$	6

Interpolating sequences

Q What is the proper way of defining $C(-\frac{1}{2})$?

EG $a(n) = n!$ is interpolated by $a(x) = \Gamma(x+1) = \displaystyle\int_0^\infty t^x e^{-t} \,\mathrm{d}t.$

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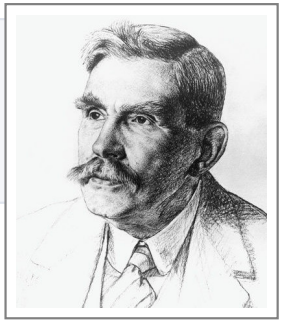
THM Glaisher 1874

$$\int_0^\infty \big(a(0) - a(1)x^2 + a(2)x^4 - \ldots\big)\,\mathrm{d}x = \frac{\pi}{2}a(-\tfrac{1}{2})$$



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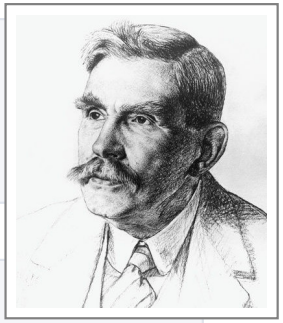
THM (Glaisher 1874)

$$\int_0^\infty \left(a(0) - a(1)x^2 + a(2)x^4 - \ldots\right) \,\mathrm{d}x = \frac{\pi}{2} a(-\tfrac{1}{2})$$



"poof"

$$\int_0^\infty \frac{1}{1+x^2 S} \cdot a(0) \,\mathrm{d}x = \frac{\pi}{2} S^{-1/2} \cdot a(0)$$

□

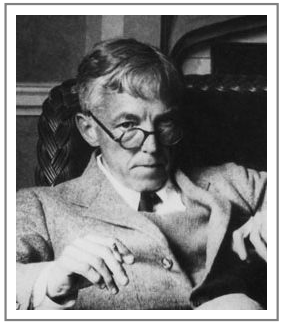
(Glaisher's formal proof, simplified by O'Kinealy)

Here, S is the shift operator: $S \cdot b(n) = b(n+1)$

Interpolating sequences: Ramanujan's master theorem

THM
Ramanujan
Hardy

$$\int_0^\infty x^{s-1}\left(a(0)-xa(1)+x^2a(2)-\ldots\right)\,\mathrm{d}x=\frac{\pi}{\sin s\pi}a(-s)$$



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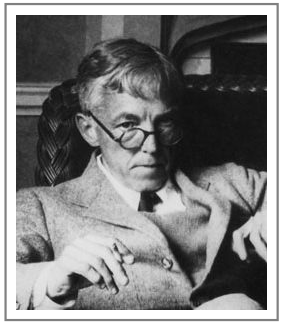
for $0<\operatorname{Re}\,s<\delta$, provided that

- a is analytic on $H(\delta)=\{z\in\mathbb{C}:\operatorname{Re}\,z\geqslant-\delta\}$,
- $|a(x+iy)|<Ce^{\alpha|x|+\beta|y|}$ for some $\beta<\pi$.



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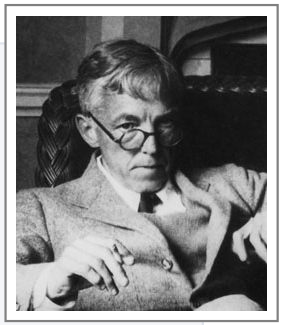
COR
Carlson
1914

Suppose a satisfies the conditions for RMT. If

$$a(0)=0,\quad a(1)=0,\quad a(2)=0,\quad \ldots,$$

then $a(z)=0$ identically.



- However, we will see that our interpolations do not arise in this way.

Interpolating sequences

Q What is the proper way of defining $C(-\frac{1}{2})$?

- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

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EG Zagier

$$(x+2)^3 A(x+2) - (2x+3)(17x^2+51x+39)A(x+1)$$
$$+ (x+1)^3 A(x) = 0 \quad \text{for all } x \in \mathbb{Z}_{\geqslant 0}$$

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In particular, $A(x)$ does not satisfy the (vertical) growth conditions of RMT.

- For the $\zeta(2)$ Apéry numbers $B(n)$, we use $B(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}$.

  However:
  - The series diverges if $\operatorname{Re} x < -1$.
  - $Q(x, S_x)B(x) = 0$ where $Q(x, S_x)$ is Apéry's recurrence operator.

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diverges for $n \notin \mathbb{Z}_{\geqslant 0}$

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$$C_{\boldsymbol{C}}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} = {}_3F_2\left(\left.\begin{matrix}-n, -n, \frac{1}{2} \\ 1, 1\end{matrix}\right| 4\right)$$

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We use the interpolation $C_{\boldsymbol{C}}(x) = \operatorname{Re}\, {}_3F_2\left(\left.\begin{matrix}-x, -x, \frac{1}{2} \\ 1, 1\end{matrix}\right| 4\right)$.

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This has a simple pole at $n = -\frac{1}{2}$.

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How to compute $C(-\frac{1}{2})$?

- RE: order 4, degree 15
- DE: order 7, degree 17
  (2 analytic solutions)

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THM McCarthy, Osburn, S 2020

For any odd prime p,

$$C(\tfrac{p-1}{2}) \equiv \gamma(p) \pmod{p^2}, \qquad \eta^{12}(2\tau) = \sum_{n\geqslant 1} \gamma(n) q^n \in S_6(\Gamma_0(4))$$

Q Is there a Zagier-type interpolation?

Beukers' proof of the irrationality of $\zeta(3)$

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n}{(1-xy)^{n+1}} \, dxdy$$

$$J_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{(1-(1-xy)w)^{n+1}} \, dxdydw$$

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- Brown realizes these as period integrals, for $N = 5, 6$, on the moduli space $\mathcal{M}_{0,N}$ of curves of genus 0 with N marked points.

Brown's cellular integrals

THM Brown 2009 — Period integrals on $\mathcal{M}_{0,N}$ are $\mathbb{Q}$-linear combinations of multiple zeta values (MZVs). (conjectured by Goncharov–Manin, 2004)

- Examples of such integrals can be written as: $(a_i, b_j, c_{ij} \in \mathbb{Z})$

$$\int_{0<t_1<\ldots<t_{N-3}<1} \prod t_i^{a_i} (1-t_j)^{b_j} (t_i - t_j)^{c_{ij}} \, dt_1 \ldots dt_{N-3}$$

- Typically involve MZVs of all weights $\leqslant N-3$.

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- Typically involve MZVs of all weights $\leqslant N-3$.
- Brown constructs families of integrals $I_\sigma(n)$, for which MZVs of submaximal weight vanish.
  Here, σ are certain ("convergent") permutations in S_N.

N	5	6	7	8	9	10	11
# of σ	1	1	5	17	105	771	7028

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

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EG
Panzer: HyperInt

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$
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- The leading coefficients of $I_\sigma(n)$ are:

$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

One of Brown's cellular integrals, cont'd

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LEM
McCarthy, Osburn, S 2020

$$A_\sigma(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i} \binom{n+k_i}{k_i}$$

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LEM McCarthy, Osburn, S 2020

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CONJ McCarthy, Osburn, S 2020

For each $N \geqslant 5$ and convergent σ_N, the leading coefficients $A_{\sigma_N}(n)$ satisfy $(p \geqslant 5)$

$$A_{\sigma_N}(mp^r) \equiv A_{\sigma_N}(mp^{r-1}) \pmod{p^{3r}}.$$

For $N = 5, 6$ these are the supercongruences proved by Beukers and Coster.

One of Brown's cellular integrals, cont'd

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LEM McCarthy, Osburn, S 2020

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THM McCarthy, Osburn, S 2020

For any odd prime p,

$$A_\sigma\Big(\frac{p-1}{2}\Big) \equiv \gamma(p) \pmod{p^2}$$

where $\eta^{12}(2\tau) = \sum_{n \geqslant 1} \gamma(n) q^n$ is the unique newform in $S_6(\Gamma_0(4))$.

Collecting some thoughts...

- Golyshev and Zagier observed that for

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad f(\tau) = \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\geqslant 1} \alpha_n q^n$$

  the known modular congruences have a continuous analog: weight 4

$$A(\tfrac{p-1}{2}) \equiv \alpha_p \pmod{p^2}, \qquad A(-\tfrac{1}{2}) = \tfrac{16}{\pi^2} L(f,2)$$

- We proved that the same phenomenon holds for:
  - all six sporadic sequences of Zagier — weight 3
  - an infinite family of leading coefficients of Brown's cellular integrals — odd weight k

- Proofs are computational and not satisfactorily uniform

  Do all of these have the same motivic explanation?
  Can Zagier's motivic approach (relying on Tate conjecture) be worked out explicitly in these cases?

- Further examples exist. What is the natural framework?

  Apéry-like sequences, CM modular forms, hypergeometric series, ...

- How to characterize the analytic interpolations abstractly?

  We used suitable binomial sums. But the interpolations are not unique! (Some grow like $\sin(\pi x)$ as $x \to i\infty$.)

- Polynomial analogs?

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`