On the representability of sequences as constant terms

International Conference on Modular Forms and q-Series

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Armin Straub

March 11, 2024

University of South Alabama

$$\frac{1}{n+1}\binom{2n}{n} = \operatorname{ct}\left[\left(\frac{1}{x}+2+x\right)^n (1-x)\right]$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \operatorname{ct}\left[\left(\frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz}\right)^n\right]$$

Slides available at:
`http://arminstraub.com/talks`

based on joint work with:

Alin Bostan
(Université Paris-Saclay)

Sergey Yurkevich
(University of Vienna)

Integrality of D-finite sequences

- A sequence is D-**finite** / P-recursive / holonomic if it satisfies a linear recurrence with polynomial coefficients.

EG The **Apéry numbers** $A(n)$ satisfy $A(0) = 1$, $A(1) = 5$ and

$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

$\zeta(3)$ is irrational!

OPEN Criterion or algorithm for classifying integrality of D-finite sequences?

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OPEN Criterion or algorithm for classifying integrality of D-finite sequences?

CONJ Christol '90: Every D-finite integer sequence of at most exponential growth is the diagonal of a rational function.

EG S 2014: The Apéry numbers are the diagonal of $\dfrac{1}{(1-x-y)(1-z-w)-xyzw}$.

Diagonals

$$\sum_{n_1,\ldots,n_d \geqslant 0} a(n_1,\ldots,n_d)\, x_1^{n_1} \cdots x_d^{n_d} \qquad\qquad \sum_{n \geqslant 0} a(n,\ldots,n)\, t^n$$

multivariate series

diagonal

EG

$$\frac{1}{1-x-y}$$

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multivariate series — diagonal

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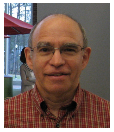
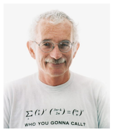
THM Gessel, Zeilberger, Lipshitz 1981–88

The diagonal of a rational function is D-finite.

More generally, the diagonal of a D-finite function is D-finite.

$F \in K[[x_1,\ldots,x_d]]$ is D-finite if its partial derivatives span a finite-dimensional vector space over $K(x_1,\ldots,x_d)$.





Characterizations of diagonals

EG Diagonals of rational functions

- $F(x)$ $=$ C-finite sequences

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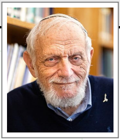

EG Diagonals of rational functions

- $F(x)$ = C-finite sequences
- $F(x, y)$ = sequences with algebraic GF (Furstenberg '67)

To see the latter, express the diagonal as $\frac{1}{2\pi i} \int_{|x|=\varepsilon} F\left(x, \frac{z}{x}\right) \frac{\mathrm{d}x}{x}$.

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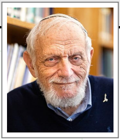

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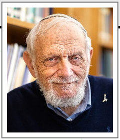
THM Bostan, Lairez, Salvy '17

Diagonals of rational functions = (multiple) binomial sums



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(i.e. $cd^n a_n \in \mathbb{Z}$ for $c, d \in \mathbb{Z}$ and at most exponential growth)

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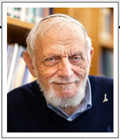

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Diagonals of rational functions over $\mathbb{Q}$ ($\subseteq$ known)
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- Open: example of a diagonal that requires more than 3 variables

Though we have numerous candidates.

The Apéry numbers as diagonals

EG
S 2014

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- OGFs of such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically leads to congruences such as

$$A(n) \equiv \begin{cases} 1 \pmod{8}, & \text{if } n \text{ even}, \\ 5 \pmod{8}, & \text{if } n \text{ odd}. \end{cases}$$

Chowla–Cowles–Cowles '80
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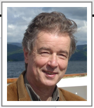
- Univariate generating function:

$$\sum_{n \geqslant 0} A(n) x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}} \, {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \right| -\frac{1024x}{(1-x+z)^4} \right),$$

where $z = \sqrt{1-34x+x^2}$.

Strands of the web of modularity



THM Beukers '87

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$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\qquad$ $q = e^{2\pi i\tau}$

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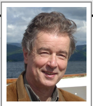

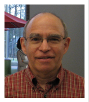

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n_i are the p-adic digits of n

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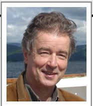
$$A\left(\frac{p-1}{2}\right) \equiv c(p) \pmod{p^2}$$

$$f(\tau) = \sum_{n\geqslant 1} c(n)q^n = \eta(2\tau)^4\eta(4\tau)^4 \in S_4(\Gamma_0(8))$$







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THM Zagier '16

$$A\left(-\tfrac{1}{2}\right) = \frac{16}{\pi^2}L(f,2)$$

- These extend to **all other** known Apéry-like numbers!!???

! = proven
? = partially known

Constant term representations

EG constant terms
$A(n) = \operatorname{ct}[L^n]$ with $L = \dfrac{(1+y)(1+z)(1+x+z)(1+x+z+yz)}{xyz}$

- $F_A(t) = \sum_{n \geqslant 0} A(n) t^n = \operatorname{ct}\left[\dfrac{1}{1-tL}\right]$ is a **period function**.

  The DE satisfied by $F_A(t)$ is the **Picard–Fuchs DE** for the family $V_t : 1 - tL = 0$.

  Generically, V_t is birationally equivalent to a **K3 surface** with Picard number 19. (Beukers–Peters '84)

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THM (Gorodetsky '21): Each sporadic sequence, except possibly (η), can be expressed as $\text{ct}[P(\boldsymbol{x})^n]$ so that the result of Samol–van Straten applies.

EG (Gorodetsky '21): (η): $\dfrac{(zx+xy-yz-x-1)(xy+yz-zx-y-1)(yz+zx-xy-z-1)}{xyz}$

$(1,0,0)$, $(1,1,0)$ and their permutations are interior points.

Constant terms

- $c(n)$ is a **constant term** if $c(n) = \operatorname{ct}[P^n(\boldsymbol{x})Q(\boldsymbol{x})]$ for Laurent polynomials $P, Q \in \mathbb{Q}[\boldsymbol{x}^{\pm 1}]$ in $\boldsymbol{x} = (x_1, \ldots, x_d)$.

EG $Q = 1$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \operatorname{ct}\left[\left(\frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz}\right)^n\right]$$

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Which diagonals are constant terms?

Which are linear combinations of constant terms?

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- We will answer this in the case of a single variable. (C-finite sequences!)
- For instance: Are Fibonacci numbers constant terms? $\dfrac{x}{1-x-x^2}$

Are Fibonacci numbers constant terms?

- Our key ingredient to answer these questions are **congruences**:

LEM Bostan, S, Yurkevich '23

If $A(n)$ is a constant term then, for all large enough primes p,

$$A(p) \equiv \underset{\in \mathbb{Q}}{\text{const}} \pmod{p}.$$

proof

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EG

The Fibonacci numbers are $F(n) = \frac{\varphi_+^n - \varphi_-^n}{\sqrt{5}}$ with $\varphi_\pm = \frac{1 \pm \sqrt{5}}{2}$.

Are Fibonacci numbers constant terms?

- Our key ingredient to answer these questions are **congruences**:

LEM Bostan, S, Yurkevich '23

If $A(n)$ is a constant term then, for all large enough primes p,

$$A(p) \equiv \underset{\in \mathbb{Q}}{\text{const}} \pmod{p}.$$

proof

$$\begin{aligned} A(p) &= \operatorname{ct}[P(\boldsymbol{x})^p Q(\boldsymbol{x})] \\ &\equiv \operatorname{ct}[P(\boldsymbol{x}^p) Q(\boldsymbol{x})] \qquad \text{(little Fermat)} \\ \text{(if } p > \deg Q) \qquad &= \operatorname{ct}[Q(\boldsymbol{x})] \operatorname{ct}[P(\boldsymbol{x}^p)] = \operatorname{ct}[Q(\boldsymbol{x})] \operatorname{ct}[P(\boldsymbol{x})] \end{aligned}$$

□

EG

The Fibonacci numbers are $F(n) = \dfrac{\varphi_+^n - \varphi_-^n}{\sqrt{5}}$ with $\varphi_\pm = \dfrac{1 \pm \sqrt{5}}{2}$.

It follows that

$$F(p) \equiv \begin{cases} 1, & \text{if } p \equiv 1, 4 \bmod 5, \\ -1, & \text{if } p \equiv 2, 3 \bmod 5, \end{cases} \pmod{p}.$$

Hence, the Fibonacci numbers cannot be constant terms.

C-finite sequences that are constant terms

DEF A sequence is C-**finite** if its generating function is rational.

- Every C-finite sequence can be represented as
$$\underset{\text{(finite support)}}{A_0(n)} + \sum_{j=1}^{d} \sum_{r=0}^{m_j - 1} c_{j,r} n^r \lambda_j^n.$$
- The **characteristic roots** are the λ_j, as well as 0 if $A_0 \not\equiv 0$.

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EG $\lambda = 2$

- $2^n = \operatorname{ct}\left[(x+2)^n\right] = \operatorname{ct}\left[2^n\right]$
- $n^2 2^n = \operatorname{ct}\left[(x+2)^n \left(\frac{8}{x^2} + \frac{2}{x}\right)\right]$

- In general: $\operatorname{ct}\left[(x+\lambda)^n \left(\frac{\lambda}{x}\right)^r\right] = \binom{n}{r} \lambda^n$

EG $\lambda = 0$

A sequence $A_0(n)$ with finite support $[0, N]$:

$$A_0(n) = \operatorname{ct}[x^n (A_0(0) + A_0(1) x^{-1} + \cdots + A_0(N) x^{-N})]$$

C-finite sequences that are constant terms, cont'd

THM Bostan, S, Yurkevich '23: There are no further C-finite sequences that are constant terms. Or linear combinations of constant terms.

- More precisely: A C-finite sequence $A(n)$ is a $\mathbb{Q}$-linear combination of r constant terms if and only if it has at most r distinct characteristic roots, all rational.
- If the $A(n)$ are integers, then the characteristic roots are integers. (Carlo Sanna '23)

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EG Fibonacci and Lucas numbers are not (sums of) constant terms.

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EG Fibonacci and Lucas numbers are not (sums of) constant terms.

EG $2^n + 1$ is not a constant term but is a sum of two.

The case of pure powers

Q Zagier '16

Which integer sequences are constant terms $\operatorname{ct}[P(\boldsymbol{x})^n]$?

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Let $A(n) = \operatorname{ct}[P(\boldsymbol{x})^n Q(\boldsymbol{x})]$ with $P, Q \in \mathbb{Q}[\boldsymbol{x}^{\pm 1}]$. TFAE:

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1. $A(n) = A(0)\,\mathrm{ct}[P(\boldsymbol{x})^n]$.
2. $A(p^r n) \equiv A(p^{r-1} n) \pmod{p^r}$ for p large enough (**Gauss congruences**)

- Gauss congruences satisfied by **realizable** sequences $a(n)$:

$$a(n) = \#\{x \in X : T^n x = x\}$$

"points of period n"

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

In fact, up to a positivity condition, the congruences characterize realizability.

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COR A C-finite sequence $A(n)$ is a constant term $\operatorname{ct}[P(\boldsymbol{x})^n]$ iff it satisfies the Lucas congruences.

$$A(n) \equiv A(n_0) A(n_1) \cdots A(n_r) \pmod{p}$$

n_i are the p-adic digits of n

In both cases, the only sequences are $A(n) = \lambda^n$.
For Lucas congruences, this was proved by McIntosh '92.

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  These are the D-finite sequences of order 1.

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EG open! — Is the following hypergeometric sequence a diagonal?

$$A(n) = \frac{\left(\frac{1}{9}\right)_n \left(\frac{4}{9}\right)_n \left(\frac{5}{9}\right)_n}{n!^2 \left(\frac{1}{3}\right)_n}$$

$3^{6n} A(n) = 1, 60, 20475, 9373650, 4881796920, \ldots$

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LEM Bostan, S, Yurkevich '23: This hypergeometric sequence is not a constant term (or a linear combination of constant terms).

Proof idea: $A(p)$ takes different values modulo p depending on whether $p \equiv \pm 1 \pmod 9$.

Constant terms are special

- For hypergeometric sequences: (or C-finite or D-finite sequences)

$$\underset{\text{(or linear combinations)}}{\{\text{constant terms}\}} \subsetneq \{\text{diagonals}\} \subseteq \{P\text{-finite \& globally bounded seq's}\}$$

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- The following is an indication that constant terms are special among diagonals and often have significant additional arithmetic properties.

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Let $A_m(n) = \dfrac{\left(\frac{1}{m}\right)_n \left(1 - \frac{1}{m}\right)_n}{n!^2}$ where $m \geqslant 2$ is an integer.

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1. $A_m(n)$ is a diagonal for all $m \geqslant 2$.
2. $A_m(n)$ is a constant term if and only if $m \in \{2, 3, 4, 6\}$.

- The cases $m \in \{2, 3, 4, 6\}$ correspond to the hypergeometric functions underlying Ramanujan's theory of elliptic functions.
  ($m = 2$: classical case; $m = 3, 4, 6$: alternative bases)

Conclusions & Outlook

- Constant terms are an arithmetically interesting subset of diagonals.
- We have classified them in the case of a single variable. Natural classes of sequences to consider next:
  - Hypergeometric sequences
  - Algebraic sequences (diagonals in two variables)
  - Algebraic hypergeometric series
  - Integral factorial ratios (Bober, 2007; via Beukers–Heckman)

EG

Is $A(n) = \frac{(8n)!n!}{(4n)!(3n)!(2n)!} = \binom{8n}{4n}\binom{4n}{n}\binom{2n}{n}^{-1}$ a constant term?

$$1, 140, 60060, 29745716, 15628090140, \ldots = \operatorname{ct}\left[\left(\frac{(1+x)^8}{(1-x)^2x^3}\right)^n\right]$$

This is algebraic (and therefore a diagonal) and hypergeometric.

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This is algebraic (and therefore a diagonal) and hypergeometric.

- How to find representations as (nice) constant terms or diagonals?

  Once found, such representations can be proved using **creative telescoping**.
- How unique are the Laurent polynomials in a constant term?

  Connections to cluster algebras, mutations of Laurent polynomials, ...

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

A. Bostan, A. Straub, S. Yurkevich
On the representability of sequences as constant terms
Journal of Number Theory, Vol. 253, 2023, p. 235–256