# Gaussian binomial coefficients with negative arguments

## International Conference on Mathematical Analysis and Applications
## National Institute of Technology Jamshedpur


**Armin Straub**

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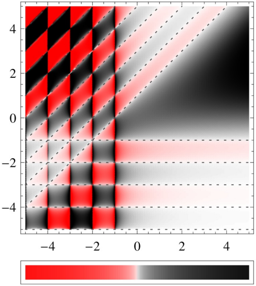
University of South Alabama


**includes joint work with:**


Sam Formichella
(University of South Alabama)

# Basic $q$-analogs

**IDEA** A $q$-analog reduces to the classical object in the limit $q \to 1$.

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For $q$-series fans:

D1

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**EG**

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^4+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

$$= (1-q+q^2) \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

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$$= \underbrace{(1-q+q^2)}_{=\Phi_6(q)} \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

$\Phi_6(1) = 1$
becomes invisible

# Cyclotomic polynomials

**DEF** The $n$th cyclotomic polynomial:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k < n \\ (k,n)=1}} (q - \zeta^k) \qquad \text{where } \zeta = e^{2\pi i/n}$$

**irreducible** polynomial (nontrivial; Gauss!) with **integer** coefficients

- $$[n]_q = \frac{q^n - 1}{q - 1} = \prod_{\substack{1 < d \leqslant n \\ d|n}} \Phi_d(q)$$

For primes: $[p]_q = \Phi_p(q)$

**EG**

$$\begin{aligned}
\Phi_5(q) &= q^4 + q^3 + q^2 + q + 1 \\
\Phi_{21}(q) &= q^{12} - q^{11} + q^9 - q^8 + q^6 - q^4 + q^3 - q + 1 \\
\Phi_{105}(q) &= q^{48} + q^{47} + q^{46} - q^{43} - q^{42} - 2q^{41} - q^{40} - q^{39} \\
&\quad + q^{36} + q^{35} + q^{34} + q^{33} + q^{32} + q^{31} - q^{28} - q^{26} - q^{24} \\
&\quad - q^{22} - q^{20} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} + q^{12} - q^9 \\
&\quad - q^8 - 2q^7 - q^6 - q^5 + q^2 + q + 1
\end{aligned}$$

# $q$-binomials: factored and expanded

**LEM factored**

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}$$

$\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor \in \{0,1\}$

**proof**

$$[n]_q! = \prod_{m=1}^{n} \prod_{\substack{d|m \\ d>1}} \Phi_d(q) = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor}$$

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**EG expanded**

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\binom{9}{3}_q = q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1$$

- The coefficients are positive and **unimodal**. Sylvester, 1878

# $q$-binomials: combinatorial

D2

**THM**

$$\binom{n}{k}_q = \sum_Y q^{w(Y)} \qquad \text{where } w(Y) = \overbrace{\sum_j y_j - j}^{\text{“normalized sum of } Y\text{”}}$$

The sum is over all $k$-element subsets $Y$ of $\{1, 2, \ldots, n\}$.

**EG**

$$\underbrace{\{1,2\}}_{\to 0}, \quad \underbrace{\{1,3\}}_{\to 1}, \quad \underbrace{\{1,4\}}_{\to 2}, \quad \underbrace{\{2,3\}}_{\to 2}, \quad \underbrace{\{2,4\}}_{\to 3}, \quad \underbrace{\{3,4\}}_{\to 4}$$

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The coefficient of $q^m$ in $\binom{n}{k}_q$ counts the number of

- $k$-element subsets of $n$ whose normalized sum is $m$,
- partitions $\lambda$ of $m$ whose Ferrer's diagram fits in a $k \times (n-k)$ box.

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$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$$

D3

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D4

# $q$-binomials: three characterizations

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**THM** Suppose $yx = qxy$ (and that $q$ commutes with $x, y$). Then:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}$$

D5

**DEF** The $q$-**derivative**:

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

**EG**

$$D_q x^n = \frac{(qx)^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1} = [n]_q \, x^{n-1}$$

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- $e_q^x \cdot e_{1/q}^{-x} = 1$

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- The $q$-**integral**:

from formally inverting $D_q$

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- The **$q$-gamma function**:

$$\Gamma_q(s) = \int_0^\infty x^{s-1} e_{1/q}^{-qx} \, \mathrm{d}_q x$$

D6

- $\Gamma_q(s+1) = [s]_q \, \Gamma_q(s)$
- $\Gamma_q(n+1) = [n]_q!$

Can similarly define **$q$-beta** via a $q$-Euler integral.

# Summary: the $q$-binomial coefficient

The $q$-binomial coefficient has a variety of natural characterizations:

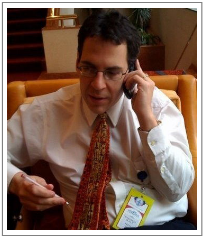
- $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$
- Via a $q$-version of **Pascal's rule**
- **Combinatorially**, as the generating function of the element sums of $k$-subsets of an $n$-set
- **Geometrically**, as the number of $k$-dimensional subspaces of $\mathbb{F}_q^n$
- **Algebraically**, via a binomial theorem for noncommuting variables
- **Analytically**, via $q$-integral representations
- Not touched here: **quantum groups** arising in representation theory and physics

# Binomial coefficients with integer entries

$$\binom{-3}{5} = -21, \qquad \binom{-3}{-5} = 6$$

$$\binom{-3.001}{-5.001} \approx 6.004$$

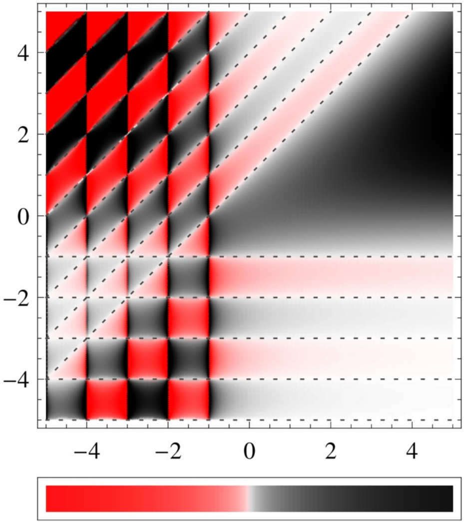
$$\binom{-3.003}{-5.005} \approx 10.03$$



1989: Ph.D. at MIT (Rota)
1996+: in mathematical finance

**Daniel E. Loeb**
*Sets with a negative number of elements*
Advances in Mathematics, Vol. 91, p.64–74, 1992

# A function in two variables

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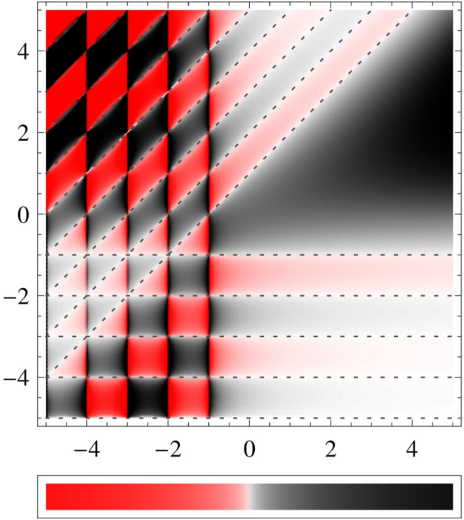


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This is a plot of:

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$$

Defined and smooth on $\mathbb{R}^2 \backslash \{x = -1, -2, \ldots\}$.

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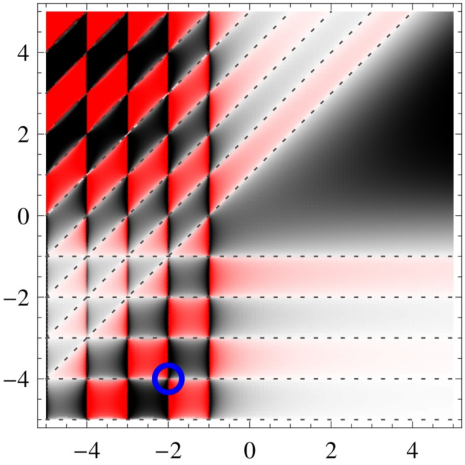


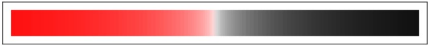

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Directional limits exist at integer points:

$$\lim_{\varepsilon \to 0} \binom{-2+\varepsilon}{-4+r\varepsilon} = \frac{1}{2!} \lim_{\varepsilon \to 0} \frac{\Gamma(-1+\varepsilon)}{\Gamma(-3+r\varepsilon)} = 3r$$

since $\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!} \frac{1}{\varepsilon} + O(1)$

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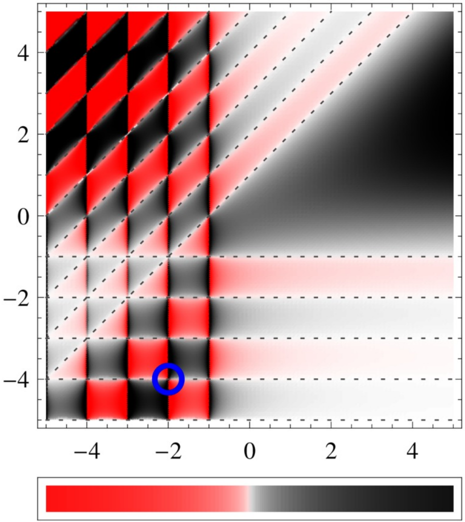


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**DEF** For all $x, y \in \mathbb{Z}$:

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# Sets with a negative number of elements

**DEF** **Hybrid sets and their subsets**

negative multiplicity

$$\{ 1, 1, 4 \mid 2, 3, 3 \}$$

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$\boldsymbol{Y} \subset \boldsymbol{X}$ if one can repeatedly **remove** elements from $X$ and thus obtain $Y$ or have removed $Y$.

**removing** $=$ decreasing the multiplicity of an element with nonzero multiplicity

**EG** Subsets of $\{1, 1, 4|2, 3, 3\}$ include:

(remove $4$) $\{4|\}$, $\{1, 1|2, 3, 3\}$

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$$\{\underbrace{1,1,4}_{\text{positive multiplicity}} \mid \overbrace{2,3,3}^{\text{negative multiplicity}}\}$$

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| | | |
|---|---|---|
| (remove $4$) | $\{4\|\}$, | $\{1,1\|2,3,3\}$ |
| (remove $4,2,2$) | $\{2,2,4\|\}$, | $\{1,1\|2,2,2,3,3\}$ |

Note that we cannot remove $4$ again. $\{4,4|\}$ is not a subset.

# Counting subsets of hybrid sets

- **New sets:** 3 elements: $\{1, 2, 4|\}$ all multiplicities $0, 1$ or $-4$ elements: $\{|1, 2, 4, 5\}$ all multiplicities $0, -1$

**THM** Loeb 1992: For all integers $n$ and $k$, the number of $k$-element subsets of an $n$-element new set is $\left|\binom{n}{k}\right|$.

**EG** A usual set like $\{1, 2, 3|\}$ only has the usual subsets.

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**EG** ($n=-3$)

- $\left|\binom{-3}{2}\right| = 6$ because the $2$-element subsets of $\{|1,2,3\}$ are:

$$\{1,1|\}, \quad \{1,2|\}, \quad \{1,3|\}, \quad \{2,2|\}, \quad \{2,3|\}, \quad \{3,3|\}$$

# Counting subsets of hybrid sets

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- $\left|\binom{-3}{-4}\right| = 3$ because the $-4$-element subsets of $\{|1,2,3\}$ are:

$$\{|1,1,2,3\},\quad \{|1,2,2,3\},\quad \{|1,2,3,3\}$$

# The binomial theorem

**THM** Loeb 1992 — For all integers $n$ and $k$, $\binom{n}{k} = \{x^k\}(1+x)^n.$

Here, we extract appropriate coefficients:

$$\{x^k\}f(x) := \begin{cases} a_k & \text{if } k \geqslant 0 \\ b_k & \text{if } k < 0 \end{cases}$$

around $x = 0$:

$$f(x) = \sum_{k \geqslant k_0} a_k x^k$$

around $x = \infty$:

$$f(x) = \sum_{k \geqslant k_0} b_{-k} x^{-k}$$

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**EG**

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 + O(x^5) \quad \text{as } x \to 0$$

$$(1+x)^{-3} = x^{-3} - 3x^{-4} + 6x^{-5} + O(x^{-6}) \quad \text{as } x \to \infty$$

Hence, for instance,

$$\binom{-3}{4} = 15, \qquad \binom{-3}{-5} = 6.$$

# $q$-binomial coefficients with integer entries

**DEF** For all integers $n$ and $k$,

$$\binom{n}{k}_q := \lim_{a \to q} \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}}.$$

$$\binom{-3}{4}_q = \frac{1}{q^{18}}(1-q+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)$$

$$\binom{-3}{-5}_q = \frac{1}{q^7}(1+q^2)(1+q+q^2)$$

**S. Formichella, A. Straub**
*Gaussian binomial coefficients with negative arguments*
Annals of Combinatorics, Vol. 23, Nr. 3, 2019, p. 725-748

# The $q$-binomial theorem

**THM** Formichella S 2019

Suppose $yx = qxy$. For $n, k \in \mathbb{Z}$, $$\binom{n}{k}_q = \{x^k y^{n-k}\}(x+y)^n.$$

Again, we extract appropriate coefficients:

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around $x = \infty$:

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# The $q$-binomial theorem

**THM** Formichella S 2019

Suppose $yx = qxy$. For $n, k \in \mathbb{Z}$, $\binom{n}{k}_q = \{x^k y^{n-k}\}(x+y)^n$.

Again, we extract appropriate coefficients:

$$\{x^k y^{n-k}\} f(x,y) := \begin{cases} a_k & \text{if } k \geqslant 0 \\ b_k & \text{if } k < 0 \end{cases}$$

around $x = 0$:

$$f(x) = \sum_{k \geqslant k_0} a_k x^k y^{n-k}$$

around $x = \infty$:

$$f(x) = \sum_{k \geqslant k_0} b_{-k} x^{-k} y^{n+k}$$

**EG**

$$\begin{aligned}(x+y)^{-1} &= y^{-1}(xy^{-1}+1)^{-1} \\ &= y^{-1} \sum_{k \geqslant 0} (-1)^k (xy^{-1})^k \\ &= \sum_{k \geqslant 0} (-1)^k q^{-k(k+1)/2} x^k y^{-k-1}\end{aligned}$$

$\binom{-1}{k}_q$

# Counting subsets of hybrid sets, $q$-version

**THM** Formichella S 2019

For all $n, k \in \mathbb{Z}$, $\binom{n}{k}_q = \varepsilon \sum_Y q^{\sigma(Y) - k(k-1)/2}$, $\varepsilon = \pm 1$.

The sum is over all $k$-element subsets $Y$ of the $n$-element set $X_n$.

$\varepsilon = 1$ if $0 \leqslant k \leqslant n$. $\varepsilon = (-1)^k$ if $n < 0 \leqslant k$. $\varepsilon = (-1)^{n-k}$ if $k \leqslant n < 0$.

$$X_n := \begin{cases} \{0, 1, \ldots, n-1|\} & \text{if } n \geqslant 0 \\ \{|-1, -2, \ldots, n\} & \text{if } n < 0 \end{cases}$$

$$\sigma(Y) := \sum_{y \in Y} M_Y(y) y$$

$M_Y(y)$ is the multiplicity of $y$ in $Y$.

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**EG** $n = -3$

The $-4$-element subsets of $X_{-3} = \{|-1, -2, -3\}$ are:

$\{|-1, -1, -2, -3\}$, $\sigma = 7$; $\{|-1, -2, -2, -3\}$, $\sigma = 8$; $\{|-1, -2, -3, -3\}$, $\sigma = 9$

Hence, $\binom{-3}{-4}_q = -(q^{-3} + q^{-2} + q^{-1})$. (subtract $\frac{k(k-1)}{2} = 10$)

# Conventions for binomial coefficients

Option advertised here:

$$\binom{n}{k} := \lim_{\varepsilon \to 0} \frac{\Gamma(n+1+\varepsilon)}{\Gamma(k+1+\varepsilon)\Gamma(n-k+1+\varepsilon)}$$

Alternative:

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$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

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- used in Mathematica (at least 9+)
- used in Maple (at least 18+)
- used in SageMath (at least 8.0+)

**EG** MMA 12

```
Binomial[-3, -5]
> 6
QBinomial[-3, -5, q]
> 0
```

Similarly, `expand(QBinomial(n,k,q))` in Maple 18 results in a division-by-zero error.

# Application: Lucas congruences

**THM** Lucas 1878

Let $p$ be prime. For integers $n, k \geqslant 0$,

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots \pmod{p},$$

where $n_i$, respectively $k_i$, are the $p$-adic digits of $n$ and $k$.

**EG**

$$\binom{19}{11} \equiv \binom{5}{4}\binom{2}{1} = 5 \cdot 2 \equiv 3 \pmod{7}$$

LHS $= 75,582$

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**EG**

$$\binom{-11}{-19} \equiv \binom{3}{2}\binom{5}{4}\binom{6}{6}\binom{6}{6}\cdots = 3 \cdot 5 \equiv 1 \pmod{7}$$

LHS $= 43,758$

Note the (infinite) 7-adic expansions:

$$-11 = 3 + 5 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \ldots$$
$$-19 = 2 + 4 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \ldots$$

# Application: $q$-Lucas congruences

**THM** Olive 1965, Désarménien 1982

Let $m \geqslant 2$ be an integer. For integers $n, k \geqslant 0$,

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'} \qquad (\operatorname{mod} \Phi_m(q)),$$

where $\begin{array}{l} n = n_0 + n'm \\ k = k_0 + k'm \end{array}$ with $n_0, k_0 \in \{0, 1, \ldots, m-1\}$.

**B. Adamczewski, J. P. Bell, and E. Delaygue.**
*Algebraic independence of $G$-functions and congruences "à la Lucas"*
*Annales Scientifiques de l'École Normale Supérieure*, 2016

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**EG**

$$\binom{-11}{-19}_q \equiv \binom{3}{2}_q \binom{-2}{-3} = -2(1+q+q^2) \pmod{\Phi_7(q)}$$

- LHS $= \frac{1}{q^{116}}(1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots + q^{80})$
- $q = 1$ reduces to $\binom{-11}{-19} \equiv -6 \equiv 1 \pmod{7}$.

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# Advertisement: More Lucas congruences

Apéry's proof of the irrationality of $\zeta(3)$ centers around:

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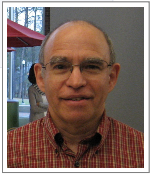
$$A(n) \equiv A(n_0)A(n_1)\cdots A(n_r) \pmod{p},$$

where $n_i$ are the $p$-adic digits of $n$.



- Gessel's approach generalized by McIntosh (1992)

**R. J. McIntosh**
*A generalization of a congruential property of Lucas.*
Amer. Math. Monthly, Vol. 99, Nr. 3, 1992, p. 231–238

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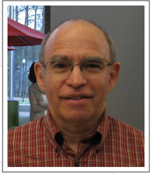

- Gessel's approach generalized by McIntosh (1992)
- $6+6+3$ sporadic Apéry-like sequences are known.

**THM** Malik–S 2015

Every (known) sporadic sequence satisfies these Lucas congruences modulo every prime.


**A. Malik, A. Straub**
*Divisibility properties of sporadic Apéry-like numbers*
Research in Number Theory, Vol. 2, Nr. 1, 2016, p. 1–26

**R. J. McIntosh**
*A generalization of a congruential property of Lucas.*
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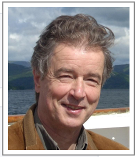
The **Apéry numbers**

$$A(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$$

satisfy many interesting properties, including **supercongruences**: $p\geqslant 5$ prime



**THM** Beukers 1985

$$A(p^r m-1)\equiv A(p^{r-1}m-1)\pmod{p^{3r}}$$

**THM** Coster 1988

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# Application: Apéry number supercongruences

The **Apéry numbers**

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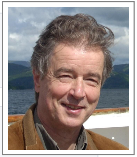

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- Extend $A(n)$ to integers $n$:

$$A(n) = \sum_{k \in \mathbb{Z}} \binom{n}{k}^2 \binom{n+k}{k}^2$$

- It then follows that:

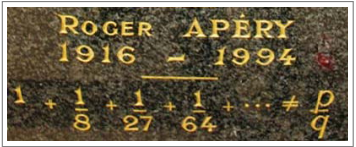
$$A(-n) = A(n-1)$$

Uniform proof (and explanation) of Beukers/Coster supercongruences

# Apéry numbers

**CONJ** $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

- Apéry (1978): $\zeta(3)$ is irrational
- Open: $\zeta(5)$ is irrational
- Open: $\zeta(3)$ is transcendental
- Open: $\zeta(3)/\pi^3$ is irrational
- Open: Catalan's constant $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is irrational




**A. Straub**
*Supercongruences for polynomial analogs of the Apéry numbers*
Proceedings of the American Mathematical Society, Vol. 147, 2019, p. 1023-1036

# Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2 + 17n + 5)u_n - n^3 u_{n-1}.$$

**THM** Apéry '78: $\zeta(3) = \sum_{n \geqslant 1} \frac{1}{n^3}$ is irrational.

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**proof** The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

# Zagier's search and Apéry-like numbers

- The Apéry numbers $B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$ for $\zeta(2)$ satisfy

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \qquad (a,b,c) = (11,3,-1).$$

| | |
|---|---|
| **Q** **Beukers** | Are there other tuples $(a,b,c)$ for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral? |

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- Apart from degenerate cases, Zagier found $6$ sporadic integer solutions:

| | | | |
|---|---|---|---|
| **A** | $\sum_{k=0}^{n} \binom{n}{k}^3$ | **D** | $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$ |
| **B** | $\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$ | **E** | $\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$ |
| **C** | $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$ | **F** | $\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$ |

# Modularity of Apéry-like numbers

- The Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left( \frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)} \right)^n}_{\text{modular function}} .$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

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**FACT** Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

| | |
|---|---|
| $f(\tau)$ | modular form of weight $k$ |
| $x(\tau)$ | modular function |
| $y(x)$ | such that $y(x(\tau)) = f(\tau)$ |

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

# Supercongruences for Apéry numbers

- Chowla, Cowles and Cowles (1980) conjectured that, for $p \geqslant 5$,
$$A(p) \equiv 5 \pmod{p^3}.$$
- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

**THM** Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$
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**EG** Simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme (1862) showed that, in fact,

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$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}$$

Ljunggren '52

# Supercongruences for Apéry-like numbers

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers. Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences related to $\zeta(3)$:

| $A(n)$ | |
|---|---|
| $\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$ | Beukers, Coster '87-'88 |
| $\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$ | Osburn–Sahu–S '16 |
| $\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$ | Osburn–Sahu '11 |
| $\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$ | open (modulo $p^3$: Amdeberhan–Tauraso '16) |
| $\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$ | Osburn–Sahu–S '16 |
| $\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$ | Gorodetsky '18 |

# Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(G)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{"points of period } n\text{"}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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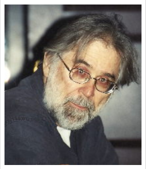
- (G) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

# $q$-congruences of Clark and Andrews



**THM** Clark 1995

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} \pmod{\Phi_n(q)^2}$$

**proof** $a=2$, $b=1$

Combinatorially, we have $q$-Chu-Vandermonde:

$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$

□

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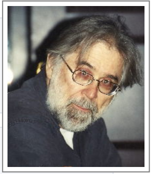

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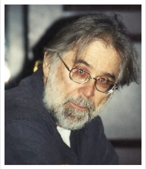
$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$
$$\equiv q^{n^2} + 1 = [2]_{q^{n^2}} \pmod{\Phi_n(q)^2}$$

(Note that $\Phi_n(q)$ divides $\binom{n}{k}_q$ unless $k=0$ or $k=n$.) □

- $\Phi_n(1) = 1$ if $n$ is not a prime power.

# $q$-congruences of Clark and Andrews



**THM** Clark 1995

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} \pmod{\Phi_n(q)^2}$$

**proof** $a = 2$, $b = 1$

Combinatorially, we have $q$-Chu-Vandermonde:

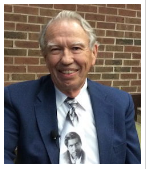
$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$

$$\equiv q^{n^2} + 1 = [2]_{q^{n^2}} \pmod{\Phi_n(q)^2}$$

(Note that $\Phi_n(q)$ divides $\binom{n}{k}_q$ unless $k = 0$ or $k = n$.) □

- $\Phi_n(1) = 1$ if $n$ is not a prime power.
- Similar results by Andrews (1999); e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \pmod{[p]_q^2}$$



# A $q$-analog of Ljunggren's congruence

- The following answers Andrews' question to find a $q$-analog of Wolstenholme's congruence.

**THM** S 2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - b(a-b)\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}$$

**EG** $n = 13$, $a = 2$, $b = 1$

$$\binom{26}{13}_q = \underbrace{1+q^{169}}_{\to 2} - \underbrace{14(q^{13}-1)^2}_{\to 0} + \underbrace{(1+q+\ldots+q^{12})^3}_{\to 13^3} f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \in \mathbb{Z}[q]$.

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- Note that $\frac{n^2-1}{24}$ is an integer if $(n,6)=1$.

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**THM** S 2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - b(a-b)\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \qquad (\mathrm{mod}\ \Phi_n(q)^3)$$

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where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \in \mathbb{Z}[q]$.

- Note that $\dfrac{n^2-1}{24}$ is an integer if $(n,6)=1$.
- $\binom{ap}{bp} \equiv \binom{a}{b}$ holds modulo $p^{3+r}$ where $r$ is the $p$-adic valuation of

$$a\, b(a-b)\binom{a}{b}.$$

Jacobsthal 1952

# A $q$-analog of Ljunggren's congruence

- The following answers Andrews' question to find a $q$-analog of Wolstenholme's congruence.

**THM** S 2011/18

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**THM** Zudilin 2019

Extension of above congruence to $q$-analog of ($p \geqslant 5$)

$$\binom{ap}{bp} \equiv \binom{a}{b} + ab(a-b)p\sum_{k=1}^{p-1}\frac{1}{k} \pmod{p^4}.$$

**Q** Creative microscoping à la Guo and Zudilin?
Extra parameter $c$ and congruences modulo, say, $\Phi_n(q)(1-cq^n)(c-q^n)$.

# A $q$-version of the Apéry numbers

- A symmetric $q$-analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

This is an explicit form of a $q$-analog of Krattenthaler, Rivoal and Zudilin (2006).

**EG** The first few values are:

$$\begin{aligned} A(0) &= 1 & A_q(0) &= 1 \\ A(1) &= 5 & A_q(1) &= 1 + 3q + q^2 \\ A(2) &= 73 & A_q(2) &= 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 \\ & & &\quad + 9q^6 + 3q^7 + q^8 \\ A(3) &= 1445 & A_q(3) &= 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 \\ & & &\quad + 117q^6 + \ldots + 3q^{17} + q^{18} \end{aligned}$$

**THM**
S
2014/18

The $q$-analog of the Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfies, for any $m \geqslant 0$,

$A_q(1) = 1 + 3q + q^2, \quad A(1) = 5$

$$A_q(mn) \equiv A_{q^{m^2}}(n) - \frac{m^2-1}{12}(q^m-1)^2 n^2 A_1(n) \pmod{\Phi_m(q)^3}.$$

**THM** S 2014/18

The $q$-analog of the Apéry numbers, defined as

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- Gorodetsky (2018) recently proved $q$-congruences implying the stronger congruences $A(p^r n) \equiv A(p^{r-1} n)$ modulo $p^{3r}$.

**Q** $q$-analog and congruences for Almkvist–Zudilin numbers?

$$\sum_k (-3)^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$$

(classical supercongruences still open)

# Multivariate supercongruences



**Q** $q$-analog and congruences for Almkvist–Zudilin numbers?

$$Z(n) = \sum_k (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$$

(classical supercongruences still open)

# Multivariate supercongruences

**Q** $q$-analog and congruences for Almkvist–Zudilin numbers?

$$Z(n) = \sum_k (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$$

(classical supercongruences still open)

**EG** **S 2014** The Almkvist–Zudilin numbers are the diagonal Taylor coefficients of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

**CONJ** **S 2014** For $p \geqslant 5$, we have the multivariate supercongruences

$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

# THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

**S. Formichella, A. Straub**
*Gaussian binomial coefficients with negative arguments*
Annals of Combinatorics, Vol. 23, Nr. 3, 2019, p. 725-748

**A. Straub**
*A $q$-analog of Ljunggren's binomial congruence*
DMTCS Proceedings: FPSAC 2011, p. 897-902

**A. Straub**
*Supercongruences for polynomial analogs of the Apéry numbers*
Proceedings of the American Mathematical Society, Vol. 147, 2019, p. 1023-1036