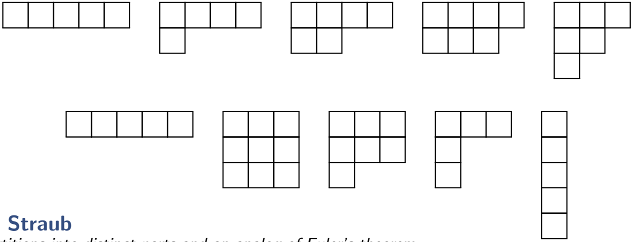
An Analog of Euler's Theorem on Integer Partitions

Mathematics Colloquium
University of South Alabama

Armin Straub

Sept 15, 2016

University of South Alabama



Armin Straub
Core partitions into distinct parts and an analog of Euler's theorem
European Journal of Combinatorics, Vol. 57, 2016, p. 40-49

Generating functions

DEF The **generating function** of a sequence $A_0, A_1, A_2, \ldots$ is

$$\sum_{n \geqslant 0} A_n x^n.$$

EG The famous **Fibonacci numbers** F_n

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

are recursively defined via

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}.$$

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Their generating function is

$$\sum_{n \geqslant 0} F_n x^n = \frac{x}{1 - x - x^2}.$$

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$$G(x) = \sum_{n \geqslant 0} F_n x^n = x + \sum_{n \geqslant 2} (F_{n-1} + F_{n-2}) x^n$$

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$$= x + xG(x) + x^2 G(x)$$

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Benefits of generating functions

We can learn a lot about a sequence from its generating function.

- closed formulas
- identities between this and other sequences
- asymptotic behaviour
- congruences
- ...

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- radius of convergence is $|\bar{\varphi}| = \varphi^{-1}$

Therefore, $\limsup_{n\to\infty} F_n^{1/n} = \varphi$.

Rational generating functions

EG **Binet**

By partial fractions,

$$\sum_{n \geqslant 0} F_n x^n = \frac{x}{1 - x - x^2} = \frac{1}{\varphi - \bar{\varphi}} \left[\frac{1}{1 - \varphi x} - \frac{1}{1 - \bar{\varphi} x} \right]$$

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$$= \frac{1}{\varphi-\bar{\varphi}}\sum_{n=0}^{\infty}(\varphi^n-\bar{\varphi}^n)x^n.$$

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$$= \frac{1}{\varphi-\bar{\varphi}} \sum_{n=0}^{\infty} (\varphi^n - \bar{\varphi}^n) x^n.$$

In other words,

$$F_n = \frac{\varphi^n - \bar{\varphi}^n}{\varphi - \bar{\varphi}} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

- This can be done for any sequence generated by a rational function. Such sequences are called **C-finite**.

Catalan numbers

Q In how many ways can a product like $abcd$ be interpreted?

EG In this case, there are five ways:

$$((ab)c)d, \quad (a(bc))d, \quad (ab)(cd), \quad a((bc)d), \quad a(b(cd))$$

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- The **Catalan number** C_n counts the the number of ways to interpret a product of $n+1$ terms. $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$

R. Stanley
Catalan Numbers
Cambridge University Press, 222 p., 2015.

Compiles 214 different objects from "combinatorics, algebra, analysis, number theory, probability theory, geometry, topology, and other areas" enumerated by C_n.

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- Write $x_0 x_1 \cdots x_{n+1}$ as $(x_0 x_1 \cdots x_k)(x_{k+1} x_{k+2} \cdots x_{n+1})$ to find:

LEM Segner

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \qquad C_0 = 1$$

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Catalan number generating function

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Solving for $F(x)$, we find that

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

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Solving for $F(x)$, we find that

$$F(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Generating function freebies

LEM

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}$$

- At a glance, we see $\limsup_{n\to\infty} C_n^{1/n} = 4$.

It is easy to be much more precise here.

EX Show that C_n also counts the number of permutations of $\{1, 2, \ldots, n\}$ that are 123-avoiding. That is, those permutations $\pi_1\pi_2\ldots\pi_n$ such that we do not have $i < j < k$ with $\pi_i < \pi_j < \pi_k$.

For instance, 2314 is not 123-avoiding because it contains 234 as a substring.

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- In particular, using Stirling's formula, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

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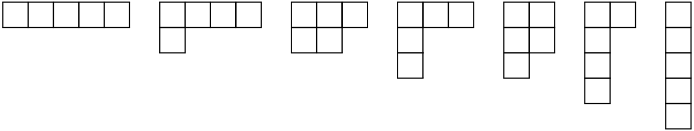
Integer partitions

- There are 7 integer partitions of 5:

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1,$$
$$2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1$$

- It is common to represent each partition by its Young diagram:



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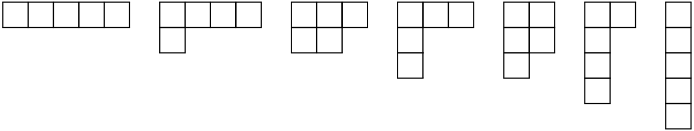

- $p(n)$ is the number of partitions of n.

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, \ldots$$

Euler's generating function

THM Euler

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k\geqslant 1} \frac{1}{1-x^k}$$

$$\begin{array}{lllll}
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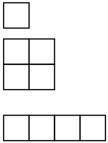

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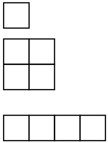

- In the same way, we have the following variations:

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$$\prod_{k \geqslant 1} \frac{1}{1-x^{2k-1}}$$

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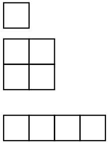

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$$\prod_{k \geqslant 1} \frac{1}{1-x^{2k-1}} = \sum_{n=0}^{\infty} p_{\text{odd}}(n) x^n$$

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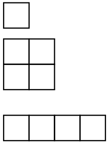

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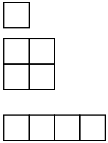

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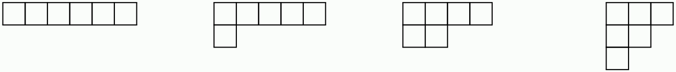
Euler's theorem

THM (Euler) The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

EG Partitions of 6 into distinct parts:

$6 \qquad 5+1 \qquad 4+2 \qquad 3+2+1$



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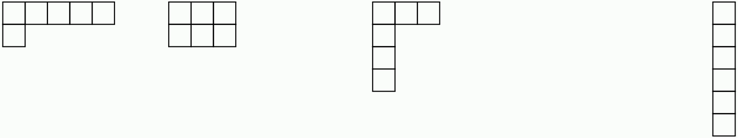

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THM Euler — The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

proof Euler famously proved his claim using a very elegant manipulation of generating functions:

$$\sum_{n=0}^{\infty} p_{\text{distinct}}(n) x^n = (1+x)(1+x^2)(1+x^3)\cdots$$

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- Bijective proofs for instance by Sylvester.

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$$= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots$$

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- Bijective proofs for instance by Sylvester.

$$p(n): 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, \ldots$$

$$p(2016) \approx 7.40 \times 10^{45}$$

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THM Hardy–Ramanujan 1918

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{2n/3}}$$

Partition congruences

THM Ramanujan 1919

$$p(5m+4) \equiv 0 \pmod{5},$$
$$p(7m+5) \equiv 0 \pmod{7},$$
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EG Atkin 1968

$$p(13 \cdot 11^3 m + 237) \equiv 0 \pmod{13}$$
$$p(17 \cdot 41^4 m + 1122838) \equiv 0 \pmod{17}$$

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$$p(13 \cdot 11^3 m + 237) \equiv 0 \pmod{13}$$
$$p(17 \cdot 41^4 m + 1122838) \equiv 0 \pmod{17}$$

- Ono (2000) and Ahlgren–Ono (2001) show that, if $\gcd(M, 6) = 1$,

$$p(Am+B) \equiv 0 \pmod{M}$$

for infinitely many non-nested arithmetic progressions $Am + B$.

CONJ No such congruences exist for moduli 2 and 3.

Partition congruences

THM Ramanujan 1919

$$p(5m+4) \equiv 0 \pmod{5},$$
$$p(7m+5) \equiv 0 \pmod{7},$$
$$p(11m+6) \equiv 0 \pmod{11}.$$

- Rank explains the congruences modulo 5 and 7. (Atkin, Swinnerton-Dyer (1954))

DEF Dyson

rank $=$ largest part $-$ number of parts

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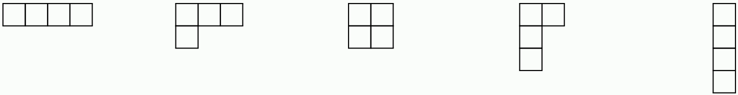

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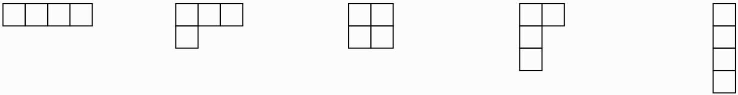

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- All three congruences are explained by Dyson's speculated **crank**, which was found by Andrews and Garvan (1988).

Modular forms

- $P(x) = \sum_{n=0}^{\infty} p(n) x^n = \prod_{k \geqslant 1} \frac{1}{1 - x^k}$ is a very special function.

DEF

$$\Delta(\tau) = \frac{q}{P(q)^{24}} = q \prod_{k \geqslant 1} (1 - q^k)^{24}, \qquad q = e^{2\pi i \tau}$$

- $\Delta(\tau + 1) = \Delta(\tau)$

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- This makes $\Delta(\tau)$ a **modular form** of weight 12 and level 1.

THM

$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\Delta(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

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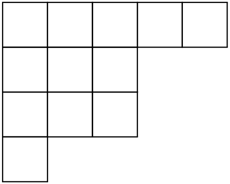
Core partitions

- The integer partition $(5, 3, 3, 1)$ has Young diagram:



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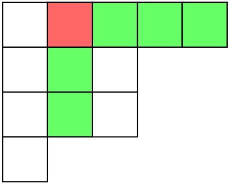

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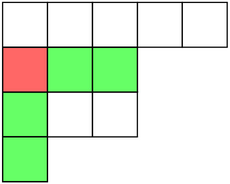

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LEM If a partition is t-core, then it is also rt-core for $r = 1, 2, 3 \ldots$

The number of core partitions

- Using the theory of modular forms, Granville and Ono (1996) showed:

(The case $t = p$ of this completed the classification of simple groups with defect zero Brauer p-blocks.)

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COR The total number of t-core partitions is infinite.

Though this is probably the most complicated way possible to see that...

Counting core partitions

THM Anderson 2002: The number of (s, t)-core partitions is finite if and only if s and t are coprime.

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- Ford, Mai and Sze (2009) show that the number of self-conjugate (s,t)-core partitions is

  $$\binom{\lfloor s/2\rfloor + \lfloor t/2\rfloor}{\lfloor s/2\rfloor}.$$

Core partitions into distinct parts

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- He further conjectured that the largest possible size of an $(s, s+1)$-core partition into distinct parts is $\lfloor s(s+1)/6 \rfloor$, and that there is a unique such largest partition unless $s \equiv 1$ modulo 3, in which case there are two partitions of maximum size.
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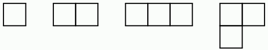

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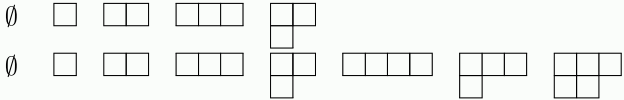
EG

$s=4$, $F_5=5$

$s=5$, $F_6=8$



A two-parameter generalization

THM
S 2016
Let $N_d(s)$ be the number of $(s, ds-1)$-core partitions into distinct parts. Then, $N_d(1) = 1$, $N_d(2) = d$ and

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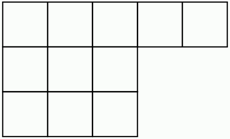
EG The first few generalized Fibonacci polynomials $N_d(s)$ are

$$1, \quad d, \quad 2d, \quad d(d+2), \quad d(3d+2), \quad d(d^2+5d+2), \ldots$$

For $d = 1$, we recover the usual Fibonacci numbers.
For $d = 2$, we find $N_2(s) = 2^{s-1}$.

The perimeter of a partition

DEF The **perimeter** of a partition is the maximum hook length in λ.

EG The partition  has perimeter 7.

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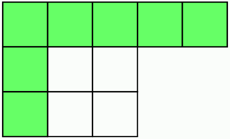
EG The partition  has perimeter 7.

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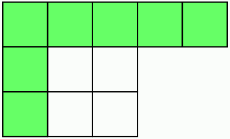
EG The partition  has perimeter 7.

- Introduced (up to a shift by 1) by Corteel and Lovejoy (2004) in their study of overpartitions.
- The perimeter is the largest part plus the number of parts (minus 1).
- The **rank** is the largest part minus the number of parts.

An analog of Euler's theorem

THM S 2016

The number of partitions into distinct parts with perimeter M equals the number of partitions into odd parts with perimeter M.

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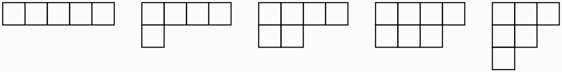

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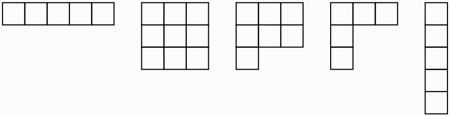

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An analog of Euler's theorem

THM
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The number of partitions into distinct parts with perimeter M equals the number of partitions into odd parts with perimeter M. Both are enumerated by the Fibonacci number F_M.

EG

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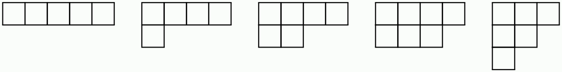

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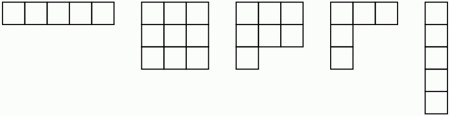

In each case, there are $F_5 = 5$ many of these partitions.

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Partitions of bounded perimeter

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COR An $(s, ds-1)$-core partition into distinct parts has perimeter at most $ds-2$.

Summary

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The number of (s,t)-core partitions is finite if and only if s and t are coprime. In that case, this number is

$$\frac{1}{s+t}\binom{s+t}{s}.$$

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Let $N_d(s)$ be the number of $(s, ds-1)$-core partitions into distinct parts. Then, $N_d(1)=1$, $N_d(2)=d$ and

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- In particular, there are F_s many $(s-1,s)$-core partitions into distinct parts,
- and 2^{s-1} many $(s,2s-1)$-core partitions into distinct parts.

Q What is the number of (s,t)-core partitions into distinct parts in general?

Enumerating (s,t)-core partitions into distinct parts

Q What is the number of (s,t)-core partitions into distinct parts?

$s \backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
9	1	5	∞	11	16	∞	64	34	∞	55	256	∞
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6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
9	1	5	∞	11	16	∞	64	34	∞	55	256	∞
10	1	∞	7	∞	∞	∞	50	∞	55	∞	89	∞
11	1	6	8	15	21	32	64	101	256	89	∞	144
12	1	∞	∞	∞	38	∞	114	∞	∞	∞	144	∞

Enumerating (s, t)-core partitions into distinct parts

Q What is the number of (s, t)-core partitions into distinct parts?

$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
9	1	5	∞	11	16	∞	64	34	∞	55	256	∞
10	1	∞	7	∞	∞	∞	50	∞	55	∞	89	∞
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3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
9	1	5	∞	11	16	∞	64	34	∞	55	256	∞
10	1	∞	7	∞	∞	∞	50	∞	55	∞	89	∞
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12	1	∞	∞	∞	38	∞	114	∞	∞	∞	144	∞

An easy exercise?

CONJ If s is odd, then the number of $(s, s+2)$-core partitions into distinct parts equals 2^{s-1}.

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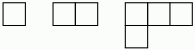
EG $(s = 3)$ The four $(3, 5)$-core partitions into distinct parts are:

$\emptyset$ 

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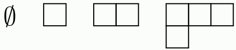

$(s=5)$ The sixteen $(5,7)$-core partitions into distinct parts are:

$$\{\}, \quad \{1\}, \quad \{2\}, \quad \{3\}, \quad \{4\}, \quad \{2,1\}, \quad \{3,1\}, \quad \{5,1\},$$
$$\{3,2\}, \quad \{4,2,1\}, \quad \{6,2,1\}, \quad \{4,3,1\}, \quad \{7,3,2\},$$
$$\{5,4,2,1\}, \quad \{8,4,3,1\}, \quad \{9,5,4,2,1\}$$

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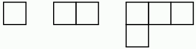
$\emptyset$ 

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$$\{3,2\},\quad \{4,2,1\},\quad \{6,2,1\},\quad \{4,3,1\},\quad \{7,3,2\},$$
$$\{5,4,2,1\},\quad \{8,4,3,1\},\quad \{9,5,4,2,1\}$$

- The largest size of such partitions appears to be $\frac{1}{384}(s^2-1)(s+3)(5s+17)$.
- There appears to be a unique partition of that size (with $\frac{1}{8}(s-1)(s+5)$ many parts and largest part $\frac{3}{8}(s^2-1)$).

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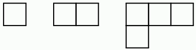
$\emptyset$ 

$(s = 5)$ The sixteen $(5, 7)$-core partitions into distinct parts are:

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- The largest size of such partitions appears to be $\frac{1}{384}(s^2-1)(s+3)(5s+17)$.
- There appears to be a unique partition of that size (with $\frac{1}{8}(s-1)(s+5)$ many parts and largest part $\frac{3}{8}(s^2-1)$).
- Yan, Qin, Jin, Zhou (2016) have very recently proven these conjectures by analyzing order ideals in an associated poset introduced by Anderson.

Enumerating (s,t)-core partitions into odd parts

Q What is the number of (s,t)-core partitions into odd parts?

$s \backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2
3	1	2	∞	4	4	∞	6	6	∞	8	8	∞
4	1	2	4	∞	7	6	9	∞	11	10	13	∞
5	1	2	4	7	∞	17	12	17	25	∞	41	31
6	1	2	∞	6	17	∞	31	21	∞	34	62	∞
7	1	2	6	9	12	31	∞	80	43	78	87	97
8	1	2	6	∞	17	21	80	∞	152	78	124	∞
9	1	2	∞	11	25	∞	43	152	∞	404	166	∞
10	1	2	8	10	∞	34	78	78	404	∞	790	308
11	1	2	8	13	41	62	87	124	166	790	∞	2140
12	1	2	∞	∞	31	∞	97	∞	∞	308	2140	∞

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$s \backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2
3	1	2	∞	4	4	∞	6	6	∞	8	8	∞
4	1	2	4	∞	7	6	9	∞	11	10	13	∞
5	1	2	4	7	∞	17	12	17	25	∞	41	31
6	1	2	∞	6	17	∞	31	21	∞	34	62	∞
7	1	2	6	9	12	31	∞	80	43	78	87	97
8	1	2	6	∞	17	21	80	∞	152	78	124	∞
9	1	2	∞	11	25	∞	43	152	∞	404	166	∞
10	1	2	8	10	∞	34	78	78	404	∞	790	308
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THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

Armin Straub
Core partitions into distinct parts and an analog of Euler's theorem
European Journal of Combinatorics, Vol. 57, 2016, p. 40-49