On a q-analog of the Apéry numbers

International conference on orthogonal polynomials and q-series
University of Central Florida

celebrating Mourad E.H. Ismail

Armin Straub

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University of Illinois at Urbana–Champaign

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

Positivity of rational functions

CONJ Kauers-Zeilberger 2008

All Taylor coefficients of the following function are positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

- Among those present, Askey, Ismail, Koornwinder have contributed to understanding the positivity of (some) rational functions.

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PROP S-Zudilin 2015

The **diagonal coefficients** of the Kauers–Zeilberger function are

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- $D(n)$ is an example of an **Apéry-like sequence**.

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Q S-Zudilin 2015 — Can we conclude the conjectured positivity from the positivity of $D(n)$ together with the (obvious) positivity of $\frac{1}{1-(x+y+z)+2xyz}$?

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

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THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q **Beukers, Zagier**: Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$${}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\right|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\left.\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\right|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2,$$

    with $\alpha=\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha=2^4,3^3,2^6,2^4\cdot 3^3$)
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Apéry-like numbers and modular forms

- The Apéry numbers $A(n)$ satisfy $\qquad 1, 5, 73, 1145, \ldots$

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4) \qquad q - 12q^2 + 66q^3 + O(q^4) \qquad q = e^{2\pi i\tau}$

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- As a consequence, with $z = \sqrt{1 - 34x + x^2}$,

$$\sum_{n \geqslant 0} A(n) x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \, {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -\frac{1024x}{(1 - x + z)^4} \right).$$

- Context:

$f(\tau)$	modular form of (integral) weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k + 1$.

Supercongruences for Apéry numbers

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EG

For primes p, simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

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$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '14
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open!! modulo p^2 Amdeberhan '14
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	Osburn–Sahu–S '14
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	open

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(C)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

Everest–van der Poorten–Puri–Ward ’02, Arias de Reyna ’05

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- $a(n) = \operatorname{tr} A^n$ with $A \in \mathbb{Z}^{d \times d}$ — Arnold '03, Zarelua '04, ...

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- $a(n) = \operatorname{tr} A^n$ with $A \in \mathbb{Z}^{d \times d}$ Arnold '03, Zarelua '04, ...

- If $a(1) = 1$, then (C) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions:

s_{10} and supercongruence known

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]$$

Cooper's sporadic sequences

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CONJ Cooper 2012

$$s_7(mp) \equiv s_7(m) \pmod{p^3} \qquad p \geqslant 3$$

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THM Osburn-Sahu-S 2014

$$s_7(mp^r) \equiv s_7(mp^{r-1}) \pmod{p^{3r}} \qquad p \geqslant 5$$

$$s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}$$

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

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- The q-factorial:

$$[n]_q! = [n]_q \, [n-1]_q \cdots [1]_q$$

- The q-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!} = \binom{n}{n-k}_q$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

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$$\begin{aligned}\binom{6}{2}_q &= \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q} \\ &= (1-q+q^2)\underbrace{(1+q+q^2)}_{=[3]_q}\underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}\end{aligned}$$

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- The cyclotomic polynomial $\Phi_6(q)$ becomes 1 for $q=1$
  and hence invisible in the classical world

The coefficients of q-binomial coefficients

- Here's some q-binomials in expanded form:

EG

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\binom{9}{3}_q = q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1$$

- The degree of the q-binomial is $k(n-k)$.
- All coefficients are positive!
- In fact, the coefficients are unimodal. Sylvester, 1878

The q-binomial coefficient $\binom{n}{k}_q$

- satisfies a q-version of Pascal's rule, $\binom{n}{j}_q = \binom{n-1}{j-1}_q + q^j \binom{n-1}{j}_q$,
- counts k-subsets of an n-set weighted by their sum,
- features in a binomial theorem for noncommuting variables,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}, \qquad \text{if } yx = qxy,$$

- has a q-integral representation analogous to the beta function,
- counts the number of k-dimensional subspaces of $\mathbb{F}_q^n$.

A q-analog of Babbage's congruence

- Combinatorially, we again obtain: "q-Chu-Vandermonde"

$$\binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q q^{(p-k)^2}$$

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$$\equiv q^{p^2} + 1 = [2]_{q^{p^2}} \qquad (\mathrm{mod}\ [p]_q^2)$$

(Note that $[p]_q$ divides $\binom{p}{k}_q$ unless $k = 0$ or $k = p$.)

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- This combinatorial argument extends to show:

THM Clark 1995

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} \pmod{[p]_q^2}$$

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- Similar results by Andrews; e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \pmod{[p]_q^2}$$

A q-analog of Ljunggren's congruence

- The following answers the question of Andrews to find a q-analog of Wolstenholme's congruence.

THM
S 2011

For any prime p,

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - (a-b)b\binom{a}{b}\frac{p^2-1}{24}(q^p-1)^2 \pmod{[p]_q^3}.$$

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EG

Choosing $p = 13$, $a = 2$, and $b = 1$, we have

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\ldots+q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132}$ is an irreducible polynomial with integer coefficients.

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  Is there a nice explanation or analog in the q-world?
- The congruence holds mod $\Phi_n(q)^3$ if p is replaced by any integer n.
  (No classical counterpart since $\Phi_n(1)=1$ unless n is a prime power.)

A q-version of the Apéry numbers

- A symmetric q-analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

- Appear implicitly in work of Krattenthaler–Rivoal–Zudilin '06

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- Appear implicitly in work of Krattenthaler–Rivoal–Zudilin '06
- The first few values are:

$$\begin{aligned} A(0) &= 1 & A_q(0) &= 1 \\ A(1) &= 5 & A_q(1) &= 1 + 3q + q^2 \\ A(2) &= 73 & A_q(2) &= 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 \\ & & &\quad + 9q^6 + 3q^7 + q^8 \\ A(3) &= 1445 & A_q(3) &= 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 \\ & & &\quad + 117q^6 + \ldots + 3q^{17} + q^{18} \end{aligned}$$

THM
S 2015
in progress

The q-Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfy the supercongruences

$$A_q(pn) \equiv A_{q^{p^2}}(n) - \frac{p^2-1}{12}(q^p-1)^2 f(n) \pmod{[p]_q^3}.$$

THM S 2015 in progress

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- The numbers $f(n)$ can be expressed as $0, 5, 292, 13005, 528016, \ldots$

$$f(n) = \sum_{k=0}^{n} g(n,k) \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad g(n,k) = k(2n-k) + \frac{k^4}{(n+k)^2}.$$

- Similar q-analogs and congruences for other Apéry-like numbers?

The Almkvist–Zudilin numbers

- Recall that for the **Almkvist–Zudilin numbers**,

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3},$$

  the supercongruences $Z(mp^r) \equiv Z(mp^{r-1})$ modulo p^{3r} are still conjectural (even for $r = 1$).

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- Maybe finding such a q-analog leads to a better understanding of the classical case, too.

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EG
S 2014

The Almkvist–Zudilin numbers are the diagonal coefficients of

$$\frac{1}{1 - (x_1 + x_2 + x_3 + x_4) + 27 x_1 x_2 x_3 x_4}.$$

Some of many open problems

- Supercongruences for all Apéry-like numbers
  - proof of all the classical ones
  - uniform explanation, proofs not relying on binomial sums
- polynomial analogs of Apéry-like numbers
  - find q-analogs (e.g., for Almkvist–Zudilin sequence)
  - q-supercongruences
  - is there a geometric picture?
- Many further questions remain.
  - is the known list complete?
  - Apéry-like numbers as diagonals and multivariate supercongruences
  - higher-order analogs, Calabi–Yau DEs
  - modular supercongruences — Beukers '87, Ahlgren–Ono '00

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \qquad \sum_{n=1}^{\infty} a(n)q^n = \eta^4(2\tau)\eta^4(4\tau)$$

  - ...

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

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Multivariate Apéry numbers and supercongruences of rational functions
Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008

R. Osburn, B. Sahu, A. Straub
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to appear in Proceedings of the Edinburgh Mathematical Society, 2014

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