Divisibility properties of sporadic Apéry-like numbers

Symbolic Computation and Special Functions
OPSFA-13, NIST

Armin Straub

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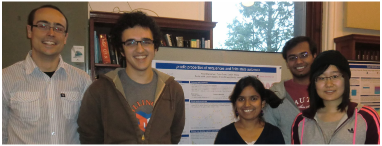
$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$



Arian Daneshvar   Amita Malik   Zhefan Wang
Pujan Dave

(Illinois Geometry Lab, UIUC, Fall 2014)

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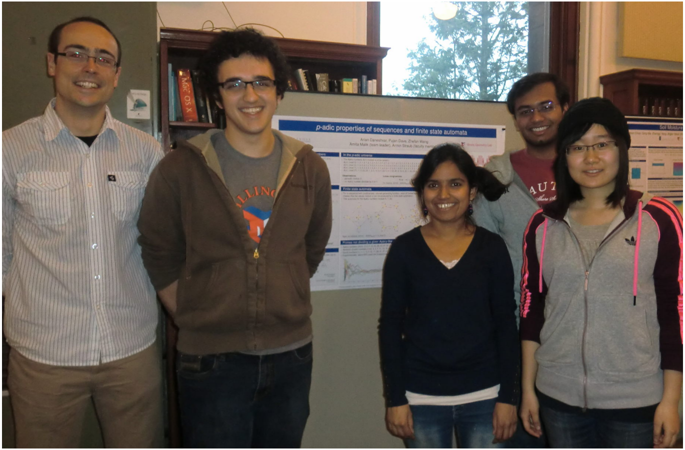

Arian Daneshvar   Amita Malik   Zhefan Wang
Pujan Dave

- semester-long project to introduce undergraduate students to research
- graduate student team leader: Amita Malik

Rough outline

- introducing Apéry-like numbers
- Lucas-type congruences
- applications
    - primes never dividing Apéry-like numbers
    - periodicity modulo p
- a little more on supercongruences (time permitting)

Positivity of rational functions

- For a smooth transition from Wadim's talk, consider the following open problem:

CONJ Kauers-Zeilberger 2008 All Taylor coefficients of the following function are positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

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PROP S-Zudilin 2015

The **diagonal coefficients** of the Kauers–Zeilberger function are

$$D(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

- $D(n)$ is an example of an **Apéry-like sequence**.

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Q S-Zudilin 2015

Can we conclude the conjectured positivity from the positivity of $D(n)$ together with the (obvious) positivity of $\frac{1}{1-(x+y+z)+2xyz}$?

Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

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THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q Beukers, Zagier: Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

Zagier's search and Apéry-like numbers

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Q (Beukers, Zagier): Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions
$$ {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\ 1,1\end{matrix}\right|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\left.\begin{matrix}\alpha,1-\alpha\\ 1\end{matrix}\right|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2,$$
  with $\alpha=\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha=2^4,3^3,2^6,2^4\cdot 3^3$)
  - 6 sporadic solutions
- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Apéry-like numbers and modular forms

- The Apéry numbers $A(n)$ satisfy $\qquad 1, 5, 73, 1145, \ldots$

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4) \qquad q - 12q^2 + 66q^3 + O(q^4) \qquad q = e^{2\pi i\tau}$

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- As a consequence, with $z = \sqrt{1 - 34x + x^2}$,

$$\sum_{n \geqslant 0} A(n) x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -\frac{1024x}{(1 - x + z)^4}\right).$$

- Context:

$f(\tau)$	modular form of (integral) weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k + 1$.

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EG

Mathematica 7 miscomputes $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ for $n > 5500$.

$$A(5 \cdot 11^3) = 12488301 \ldots \text{about 2000 digits} \ldots \text{about 8000 digits} \ldots 7956522125$$

Weirdly, with this wrong value, one still has

$$A(5 \cdot 11^3) \equiv A(5 \cdot 11^2) \pmod{11^6}.$$

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- The congruences $a(mp^r) \equiv a(mp^{r-1})$ modulo p^r occur frequently:
  - $a(n) = \operatorname{tr} A^n$ with $A \in \mathbb{Z}^{d \times d}$ Arnold '03, Zarelua '04, ...

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- The congruences $a(mp^r) \equiv a(mp^{r-1})$ modulo p^r occur frequently:
  - $a(n) = \operatorname{tr} A^n$ with $A \in \mathbb{Z}^{d \times d}$ — Arnold '03, Zarelua '04, ...
  - **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{"points of period } n\text{"}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers. Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
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$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '14
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open!! modulo p^2 Amdeberhan '14
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	Osburn–Sahu–S '14
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Lucas congruences

- Lucas showed that the beautiful congruences

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_r}{k_r} \pmod{p},$$

where n_i, respectively k_i, are the p-adic digits of n and k.

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- The Apéry numbers $A(n)$ satisfy the **Lucas congruences** (Gessel 1982)

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THM Malik–S 2015: Every sporadic sequence satisfies these Lucas congruences modulo every prime.

Primes not dividing Apéry numbers

EG

- The values of Apéry numbers $A(0), A(1), \ldots, A(6)$ modulo 7 are $1, 5, 3, 3, 3, 5, 1$.

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- Likewise, $2, 3, 7, 13, 23, 29, 43, 47, \ldots$ do not divide any Apéry number.

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- The primes below 100 not dividing sporadic sequences, as well as the proportion of primes below $10,000$ not dividing any term

(δ)	2, 5, 7, 11, 13, 19, 29, 41, 47, 61, 67, 71, 73, 89, 97	0.6192
(η)	2, 3, 17, 19, 23, 31, 47, 53, 61	0.2897
(α)	3, 5, 13, 17, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 83, 89	0.5989
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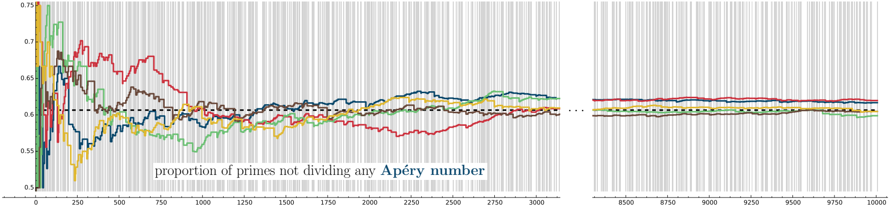

CONJ DDMSW 2015 The proportion of primes not dividing any Apéry number $A(n)$ is $e^{-1/2} \approx 60.65\%$.

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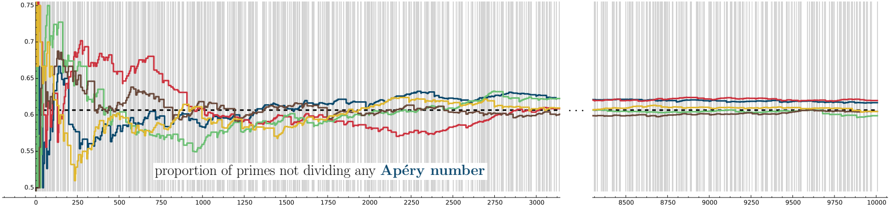

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- Heuristically, combine Lucas congruences,
- palindromic behavior of Apéry numbers, that is

$$A(n) \equiv A(p-1-n) \pmod{p},$$

- and $e^{-1/2} = \lim_{p\to\infty} \left(1 - \frac{1}{p}\right)^{(p+1)/2}$.

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CONJ Malik–S 2015

For all primes $p \neq 3$ of the form $3x^2 - 3xy + 7y^2$, with $x, y \in \mathbb{Z}_{\geqslant 0}$,

$$A_\eta\left(\frac{p-1}{3}\right) \equiv 0 \pmod{p}.$$

- Together with another such congruence, we expect the proportion of primes not dividing any $A_\eta(n)$ to be $\frac{1}{2}e^{-1/2} \approx 30.33\%$.

THM Ahlgren–Ono '00

The Apéry numbers satisfy

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}$$

with

$$\sum_{n=1}^{\infty} a(n) q^n = \eta^4(2\tau)\eta^4(4\tau).$$

- conjectured by Beukers '87, and proved modulo p
- similar congruences modulo p for other Apéry-like numbers

Periodicity of residues

THM Gessel '82

$$A(n) \equiv \begin{cases} 1 \pmod{8}, & \text{if } n \text{ even,} \\ 5 \pmod{8}, & \text{if } n \text{ odd.} \end{cases}$$

- conjectured by Chowla–Cowles–Cowles '80
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PROP Gessel '82 If $C(n)$ satisfies Lucas congruences modulo p and is eventually periodic modulo p, then

$$C(n) \equiv C(1)^n \pmod p \qquad \text{for all } n = 0, 1, \ldots, p-1.$$

EG For the Almkvist–Zudilin sequence

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3},$$

$Z(3) - Z(1)^3 = 24.$ So can be periodic only modulo 2 and 3.

THM DDMSW 2015

The Almkvist–Zudilin numbers

$$Z(n)=\sum_{k=0}^{n}(-3)^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$$

satisfy the congruences

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- This can be proved using computer algebra in two steps.

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LEM S 2014 The Almkvist–Zudilin numbers are the diagonal coefficients of

$$\frac{1}{1 - (x_1 + x_2 + x_3 + x_4) + 27 x_1 x_2 x_3 x_4}.$$

That is, $Z(n)$ equals the coefficient of $(x_1 x_2 x_3 x_4)^n$.

Diagonals

- The **diagonal** of a multivariate series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

  is the univariate function $\sum_{n \geqslant 0} a(n, \ldots, n) x^n$.

- The diagonal of an algebraic function is D-finite.

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$$\frac{1}{1 - x - y}$$

has diagonal coefficients $\binom{2n}{n}$.

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$$\frac{1}{1-x-y} = \sum_{n=0}^{\infty} (x+y)^n$$

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$$F(x_1,\ldots,x_d) = \sum_{n_1,\ldots,n_d \geqslant 0} a(n_1,\ldots,n_d) x_1^{n_1} \cdots x_d^{n_d},$$

is the univariate function $\sum_{n\geqslant 0} a(n,\ldots,n) x^n$.

- The diagonal of an algebraic function is D-finite.

Gessel
Zeilberger
Lipshitz

EG

$$\frac{1}{1-x-y} = \sum_{n=0}^{\infty} (x+y)^n$$

has diagonal coefficients $\binom{2n}{n}$. The diagonal is

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$

Finite state automata

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- The diagonal of a rational function $F(x,y)$ is always algebraic.
  To see this, express the diagonal as $\frac{1}{2\pi i} \int_{|x|=\varepsilon} F(x, \frac{y}{x}) \frac{\mathrm{d}x}{x}$.
- Not true for more than two variables.

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  To see this, express the diagonal as $\frac{1}{2\pi i} \int_{|x|=\varepsilon} F(x, \frac{y}{x}) \frac{\mathrm{d}x}{x}$.
- Not true for more than two variables. However:
  (Furstenberg '67, Deligne '84 and Denef–Lipshitz '87)

THM Diagonals of algebraic functions in $\mathbb{Z}_p[[x_1, \ldots, x_d]]$ are algebraic over $\mathbb{Z}_p(x)$. Equivalently, the diagonal coefficients modulo p^r are generated by a finite state automaton.

Finite state automata

- Recall: the AZ numbers $(-1)^n Z(n)$ are the diagonal coefficients of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4}.$$

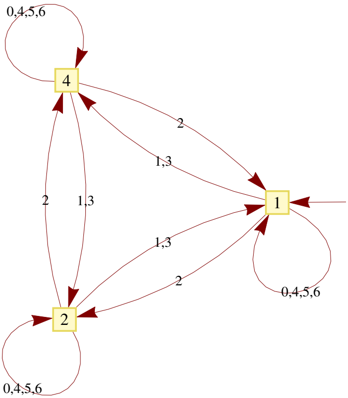
Finite state automata

- Recall: the AZ numbers $(-1)^n Z(n)$ are the diagonal coefficients of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4}.$$

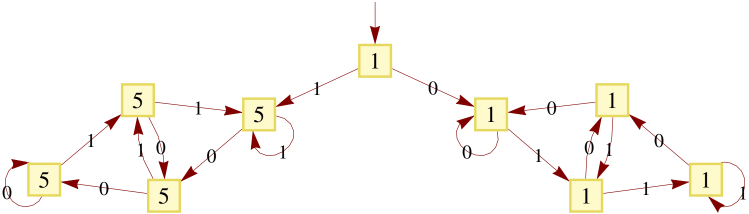
- Rowland–Yassawi (2013) give an algorithm to compute a finite state automaton for these numbers modulo 7 (or any p^r).
- For instance:
$Z(63) = Z(120_{\text{base }7}) \equiv 1 \pmod{7}$.

Of course, modulo primes it is easier to use Lucas congruences.



- The Almkvist–Zudilin numbers $Z(n)$ modulo 8:

This automatically generated automaton can, of course, be simplified.



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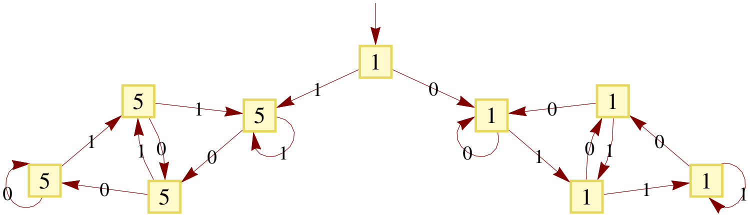

- The automaton makes it obvious that, indeed,

$$Z(n) \equiv \begin{cases} 1 \pmod{8}, & \text{if } n \text{ even,} \\ 5 \pmod{8}, & \text{if } n \text{ odd.} \end{cases}$$

Apéry numbers as diagonals

CONJ Christol 1990 — Every holonomic integer sequence with at most exponential growth is the diagonal of a rational function.

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THM S 2014 — The Apéry numbers are the diagonal coefficients of

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- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}} \, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\right| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

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- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson

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- Let $A(\boldsymbol{n})$ be the coefficient of $\boldsymbol{x}^{\boldsymbol{n}} = x_1^{n_1} \cdots x_4^{n_4}$.

  Then, for $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

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  Then, for $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Numerical evidence suggests the same congruences for

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4}.$$

Some of many open problems

- Supercongruences for all Apéry-like numbers
  - proof of all the classical ones
  - uniform explanation, proofs not relying on binomial sums
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- polynomial analogs of Apéry-like numbers
  - find q-analogs (e.g., for Almkvist–Zudilin sequence)
  - q-supercongruences
  - is there a geometric picture?
- Many further questions remain.
  - is the known list complete?
  - Apéry-like numbers as diagonals and multivariate supercongruences
  - higher-order analogs, Calabi–Yau DEs
  - modular supercongruences Beukers '87, Ahlgren–Ono '00

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \qquad \sum_{n=1}^{\infty} a(n) q^n = \eta^4(2\tau)\eta^4(4\tau)$$

  - ...

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

A. Malik, A. Straub
Divisibility properties of sporadic Apéry-like numbers
Preprint, 2015

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Multivariate Apéry numbers and supercongruences of rational functions
Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008

R. Osburn, B. Sahu, A. Straub
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to appear in Proceedings of the Edinburgh Mathematical Society, 2014

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Positivity of rational functions and their diagonals
Journal of Approximation Theory (special issue dedicated to Richard Askey), Vol. 195, 2015, p. 57-69