Positivity of rational functions and their diagonals

Special Functions and Their Applications
AMS Fall Eastern Sectional Meeting, Halifax

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Based on joint work with:

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University of Newcastle

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

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EG Szegő 1933 $\frac{1}{(1-x)(1-y)+(1-y)(1-z)+(1-z)(1-x)}$ is positive.

- Szegő's proof builds on an integral of a product of Bessel functions.
  "the used tools, however, are disproportionate to the simplicity of the statement"
- Elementary proof by Kaluza ('33)
- Askey–Gasper ('72) use integral of product of Legendre functions.
- Ismail–Tamhankar ('79) systematize Kaluza's approach by using MacMahon's Master Theorem.

Historical motivation

$$\frac{1}{(1-x)(1-y)+(1-y)(1-z)+(1-z)(1-x)} = \sum_{k,m,n} A(k,m,n) x^k y^m z^n$$

- Friedrichs and Lewy conjectured positivity of $A(k,m,n)$.
- Wanted to show convergence of finite difference approximations to

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\frac{\partial}{\partial z} \right) u(x,y,z) = 0,$$

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- With $\partial/\partial x$ replaced by Δ_k, $\Delta a(k) = a(k) - a(k-1)$

$$(\Delta_k \Delta_m + \Delta_k \Delta_n + \Delta_m \Delta_n) A(k,m,n) = 0.$$

Generalizations

- Szegő also showed that

$$\frac{1}{\sum_{i=1}^4 \prod_{j\neq i}(1-x_j)} = \frac{1}{(1-x_2)(1-x_3)(1-x_4)+\cdots+(1-x_1)(1-x_2)(1-x_3)}$$

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- Non-negativity proved by a very general result of Scott–Sokal ('13):
  - $\frac{1}{\det\left(\sum(1-x_i)A_i\right)}$ is non-negative if $A_i \geqslant 0$ are hermitian matrices.
  - For the Lewy–Askey problem:

$$A_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad A_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \quad A_3 = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \quad A_4 = \begin{bmatrix}1 & e^{-i\pi/3}\\ e^{i\pi/3} & 1\end{bmatrix}.$$

Preserving positivity

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Positivity of the Askey–Gasper rational function

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S '08
Kauers–Zeilberger '08

$$T_p(F) = \frac{F\left(\frac{px_1}{1-(1-p)x_1},\ldots,\frac{px_n}{1-(1-p)x_n}\right)}{(1-(1-p)x_1)\cdots(1-(1-p)x_n)}.$$

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- If $F(x_1, \ldots, x_n)$ is positive, so is, for $0 \leqslant p \leqslant 1$,

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EG

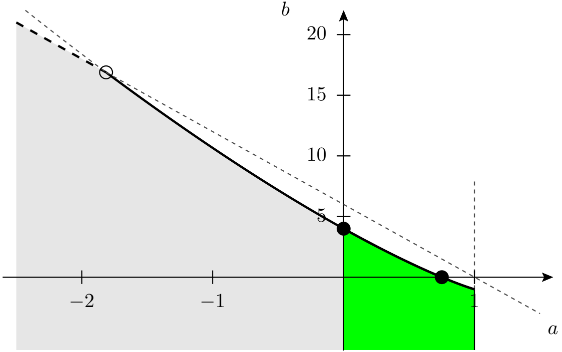
$$T_{1/2}\,\frac{1}{1-(x+y+z)+4xyz} = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

The case of three variables

$$h_{a,b}(x,y,z) = \frac{1}{1-(x+y+z)+a(xy+yz+zx)+bxyz}$$

CONJ S '08

$h_{a,b}$ is positive $\iff \begin{cases} b < 6(1-a) \\ b \leqslant 2-3a+2(1-a)^{3/2} \\ a \leqslant 1 \end{cases}$



- $h_{a,b}$ is positive in the green region — S '08
- The conditions in the conjecture are necessary for positivity — S–Zudilin '13

A conjecture of Gillis, Reznick and Zeilberger

CONJ G-R-Z '83 For any $d \geqslant 4$, the following function is non-negative:

$$\frac{1}{1-(x_1+x_2+\ldots+x_d)+d!x_1x_2\cdots x_d}$$

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- Diagonal coefficients eventually positive if $c < (d-1)^{d-1}$?

A conjecture and its diagonal

CONJ Kauers-Zeilberger 2008

The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

- Would imply conjectured positivity of Lewy–Askey rational function

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

Recent proof of non-negativity by Scott and Sokal, 2013

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PROP S-Zudilin 2013

The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

A question

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

Szegő's rational function, revisited

- Recall Szegő's rational function

$$S(x,y,z)=\frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}.$$

$S(2x,2y,2z)$ has diagonal coefficients

$$s_n=\sum_{k=0}^{n}(-27)^{n-k}2^{2k-n}\frac{(3k)!}{k!^3}\binom{k}{n-k},$$

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$$y(z)={}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle|27z(2-27z)\right).$$

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- Ramanujan's cubic transformation

$${}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle| 1-\left(\frac{1-x}{1+2x}\right)^3\right) = (1+2x)\,{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle| x^3\right),$$

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puts this in the form

$$y(z)=(1+2x(z)){}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\right|x(z)^3\right),$$

where the algebraic $x(z)=c_1z+c_2z^2+\ldots$ has positive coefficients.

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THM
S-Zudilin
2013

$$F(x,y) = \frac{1}{1 + c_1(x+y) + c_2 xy} \quad \text{is positive}$$

$\Longleftrightarrow$ diagonal of F and $F|_{x_d=0}$ are positive

Another positivity conjecture

CONJ **Kauers 2007** The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+\frac{64}{27}(yzw+xzw+xyw+xyz)}.$$

- The diagonal is positive. S–Zudilin '13
  (apply CAD to recurrence of order 3 and degree 6)
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  (because $64/27<4$)

Arithmetically interesting positive diagonals

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EG Koornwinder's rational function

$$\frac{1}{1-(x+y+z+w)+4e_3(x,y,z,w)-16xyzw}$$

has diagonal coefficients $\displaystyle\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2.$

Using a positivity preserving operator, implies positivity of

$$1/e_3(1-x,1-y,1-z,1-w)$$

Apéry-like numbers

- These last two sequences are examples of Apéry-like numbers.
- The **Apéry numbers** are $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

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- Essentially, only 14 tuples (a, b, c) found. (Almkvist–Zudilin)
- Similarly: $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)

- 4 hypergeometric and Legendrian solutions with generating functions

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\middle|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

with $\alpha = \frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4\cdot 3^3$.

- Six sporadic solutions:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k (-1)^k\binom{n}{k}^3\left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l}\binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$

Properties of Apéry-like numbers

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\quad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\quad$ $q = e^{2\pi i \tau}$

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$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\quad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\quad$ $q = e^{2\pi i \tau}$

- For $p \geqslant 5$, satisfy the **supercongruence** Beukers, Coster '85, '88

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\quad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\quad$ $q = e^{2\pi i\tau}$

- For $p \geqslant 5$, satisfy the **supercongruence** Beukers, Coster '85, '88

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

- Both properties hold for other Apéry-like numbers as well!

But some of the supercongruences are still open.

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2014

A. Straub
Positivity of Szegö's rational function
Advances in Applied Mathematics, Vol. 41, Issue 2, Aug 2008, p. 255-264