On the ubiquity of modular forms and Apéry-like numbers

Algebra & Number Theory Seminar
University College Dublin

Armin Straub

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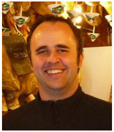
University of Illinois at Urbana–Champaign & Max-Planck-Institut für Mathematik, Bonn

Includes joint work with:



Jon Borwein, James Wan, Wadim Zudilin
University of Newcastle, Australia

Mathew Rogers
University of Montreal

INTRODUCTION

Apéry-like numbers and modular forms

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} - (2n+1)(17n^2+17n+5)u_n + n^3 u_{n-1} = 0.$$

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THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right)$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. $\square$

- Recurrence for the Apéry numbers is the case $(a,b,c)=(17,5,1)$ of

$$(n+1)^3 u_{n+1} - (2n+1)(an^2+an+b)u_n + cn^3 u_{n-1} = 0.$$

Q Are there other triples for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

Apéry-like numbers

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Q Are there other triples for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Almkvist and Zudilin find 14 triplets (a, b, c).
  The simpler case of $(n+1)^2 u_{n+1} - (an^2 + an + b)u_n + cn^2 u_{n-1} = 0$ was similarly investigated by Beukers and Zagier.
- 4 hypergeometric, 4 Legendrian and 6 sporadic solutions

- Hypergeometric and Legendrian solutions have generating functions

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\ 1,1\end{matrix}\middle|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\ 1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

  with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$.

- The six sporadic solutions are:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3\left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$

Modular functions

> *Modular forms are functions on the complex plane that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents. But they do exist.*
>
> **Barry Mazur** (BBC Interview, "The Proof", 1997)

DEF $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on the upper half-plane $\mathbb{H}$ by $\gamma \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$.

DEF $f : \mathbb{H} \to \mathbb{C}$ is a **modular function** for $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$ if

- $f(\gamma \cdot \tau) = f(\tau)$ for all $\gamma \in \Gamma$,
- f is meromorphic (including at the cusps). e.g., at $i\infty$

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EG

$$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \qquad S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

$$f(\tau + 1) = f(\tau) \qquad f(-1/\tau) = f(\tau)$$

T and S generate $\mathrm{SL}_2(\mathbb{Z})$.

Modular functions

- Equivalently, modular functions are meromorphic functions on the compactification $X(\Gamma)$ of $\mathbb{H}/\Gamma$.

EG If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then $X(\Gamma) \cong P^1(\mathbb{C})$.

$$\{\text{modular functions}\} = \mathbb{C}(j)$$

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots$$

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$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots$$

EG

$$j(i) = 1728$$

$$j\left(\tfrac{1+i\sqrt{35}}{2}\right) = -16^3(15 + 7\sqrt{5})^3$$

- In fact, for any modular function f,

$$\tau \in \mathbb{Q}(\sqrt{-d}) \quad \Longrightarrow \quad f(\tau) \in \overline{\mathbb{Q}}.$$

Modular forms

> *There's a saying attributed to Eichler that there are five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms.*
>
> **Andrew Wiles** (BBC Interview, "The Proof", 1997)

DEF A function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of weight k if

- $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^k f(\tau)$, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,
- f is holomorphic (including at the cusp $i\infty$).

EG

$$f(\tau + 1) = f(\tau), \qquad \tau^{-k} f(-1/\tau) = f(\tau).$$

- Similarly, MFs w.r.t. finite-index $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$
- Spaces of MFs finite dimensional, Hecke operators, ...

Modular forms: a prototypical example

- The **Dedekind eta function** $(q = e^{2\pi i\tau})$

$$\eta(\tau) = q^{1/24} \prod_{n \geqslant 1} (1 - q^n)$$

transforms as

$$\eta(\tau + 1) = e^{\pi i/12} \eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau).$$

EG $\eta(\tau)^{24}$ is a modular form of weight 12.

EG

$$\eta(i) = \frac{1}{2\pi^{3/4}} \Gamma\left(\frac{1}{4}\right)$$

Modularity of Apéry-like numbers

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)}_{\text{modular function}}^{12n}.$$

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FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

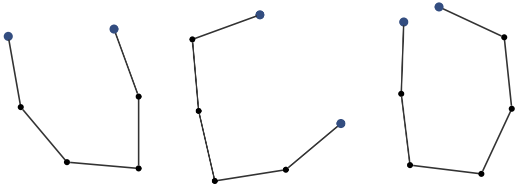
- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

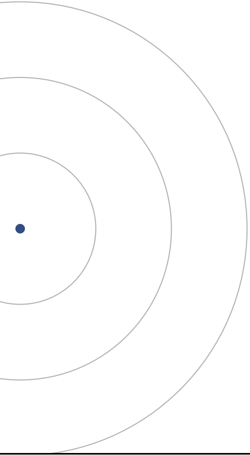
Then $y(x)$ satisfies a linear differential equation of order $k+1$.

EXAMPLE I

Short random walks



- n steps in the plane (length 1, random direction)



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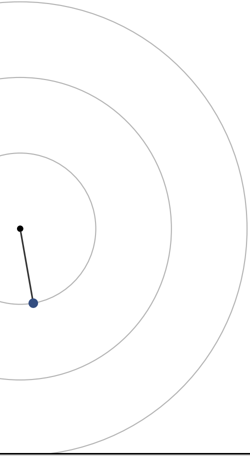

Personal encounter in the wild I: Random walks

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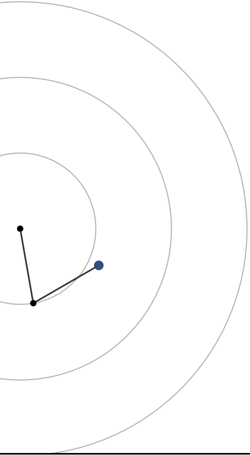

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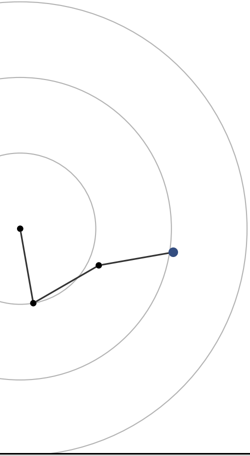

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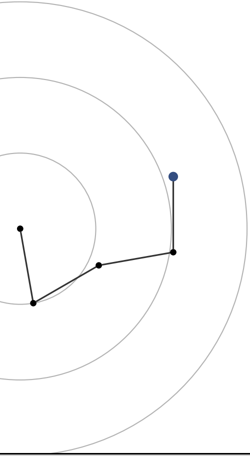

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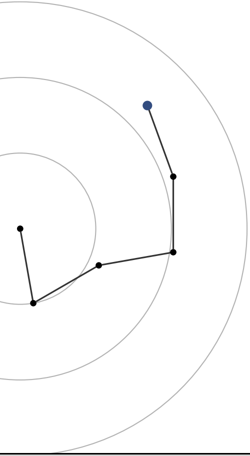

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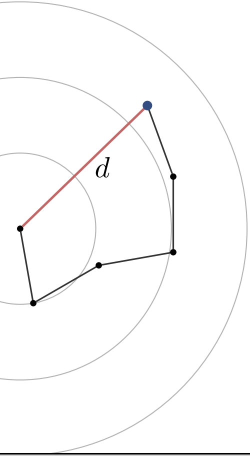
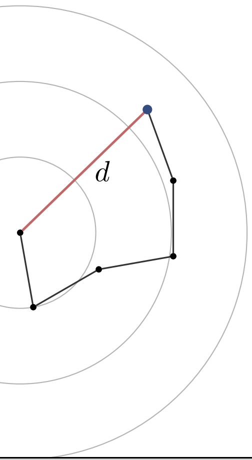
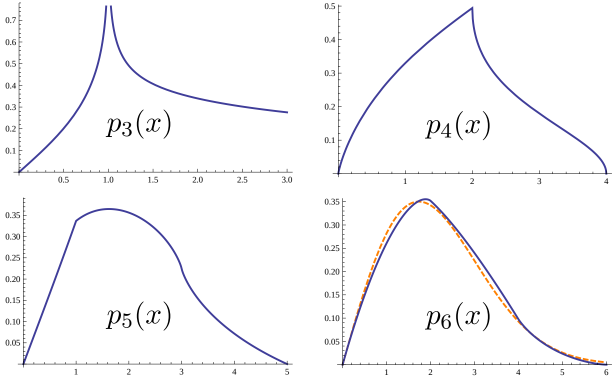


- n steps in the plane (length 1, random direction)
- $p_n(x)$: probability density of distance traveled





- The probability moments

$$W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$$

include the Apéry-like numbers

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j},$$

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}.$$

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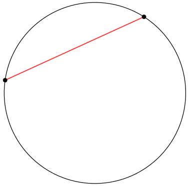
THM Borwein-Nuyens-S-Wan 2010

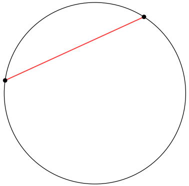
$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

Personal encounter in the wild I: Random walks

- In particular, $W_2(2k) = \binom{2k}{k}$.
- The average distance traveled in two steps is

$$W_2(1) = \binom{1}{1/2} = \frac{4}{\pi}.$$





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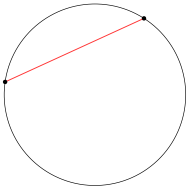
Personal encounter in the wild I: Random walks



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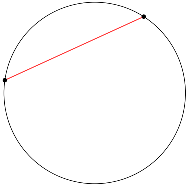
$${}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 1, 1\end{matrix}\right| 4\right) \approx 1.574597238 - 0.126026522i$$

Personal encounter in the wild I: Random walks



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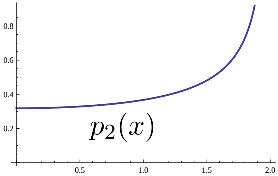
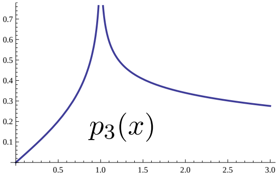
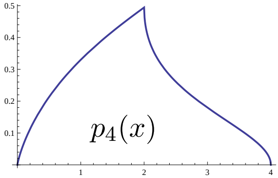
THM Borwein-Nuyens-S-Wan, 2010

$$W_3(1) = \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4} \frac{2^{2/3}}{\pi^4} \Gamma^6\left(\frac{2}{3}\right)$$
$$= 1.57459723755189\ldots$$





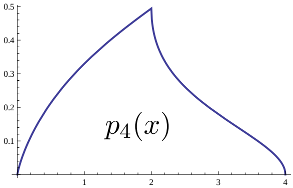


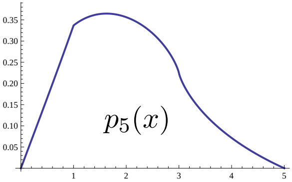
$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{x}{(3+x^2)} {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| \frac{x^2 (9-x^2)^2}{(3+x^2)^3} \right)$$ classical with a spin

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \operatorname{Re} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{(16-x^2)^3}{108x^4} \right)$$ new BSWZ 2011

Personal encounter in the wild I: Random walks





$$p_5'(0) = p_4(1) \approx 0.32993$$

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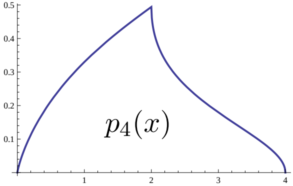

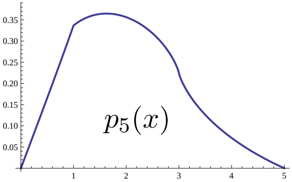

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THM **Borwein-S-Wan-Zudilin 2011**

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\left(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{\text{modular function}}\right) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{\text{modular form}}$$

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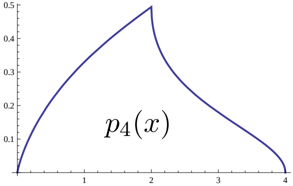

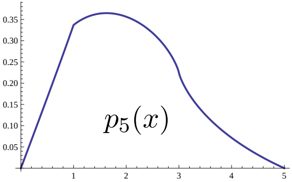

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- When $\tau = -\frac{1}{2} + \frac{1}{6}\sqrt{-15}$, one obtains $p_4(1)$ as an eta-product.
- Modular equations and Chowla–Selberg lead to:

$$p_4(1) = \frac{\sqrt{5}}{40\pi^4}\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

EXAMPLE II

Binomial congruences

$$\binom{2p-1}{p-1} \equiv 1 \mod p^3$$

Personal encounter in the wild II: Binomial congruences

John Wilson (1773, Lagrange):	$(p-1)! \equiv -1 \mod p$	
Charles Babbage (1819):	$\binom{2p-1}{p-1} \equiv 1 \mod p^2$	
Joseph Wolstenholme (1862):	$\binom{2p-1}{p-1} \equiv 1 \mod p^3$	
James W.L. Glaisher (1900):	$\binom{mp-1}{p-1} \equiv 1 \mod p^3$	
Wilhelm Ljunggren (1952):	$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$	

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THM Clark, Andrews '95–'99

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}}, \qquad \binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \mod [p]_q^2$$

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John Wilson (1773, Lagrange): $(p-1)! \equiv -1 \mod p$

Charles Babbage (1819): $\binom{2p-1}{p-1} \equiv 1 \mod p^2$

Joseph Wolstenholme (1862): $\binom{2p-1}{p-1} \equiv 1 \mod p^3$

James W.L. Glaisher (1900): $\binom{mp-1}{p-1} \equiv 1 \mod p^3$

Wilhelm Ljunggren (1952): $\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$

THM S 2011, $p \geqslant 5$

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

Personal encounter in the wild II: Binomial congruences

- Wolstenholme's congruence is the $m = 1$ case of:
  The sequence $A(n) = \binom{2n}{n}$ satisfies the **supercongruence** $(p \geqslant 5)$

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Q	How does the q-side of supercongruences for Apéry-like numbers look like?

EXAMPLE III

Ramanujan-type series for $1/\pi$

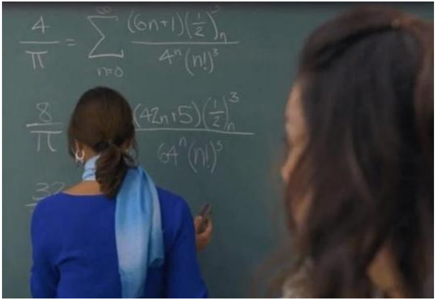
$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

Personal encounter in the wild III: Series for $1/\pi$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$



- Starred in High School Musical, a 2006 Disney production

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

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$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{\mathbf{16}}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

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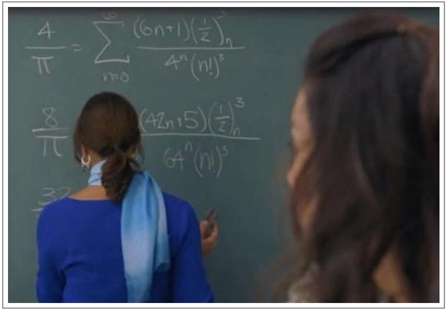

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Personal encounter in the wild III: Series for $1/\pi$

EG
Gosper 1985

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}$$

EG
Chudnovsky's 1988

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(3n)! n!^3} \frac{13591409 + 545140134n}{640320^{3n+3/2}}$$

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- The following series was conjectured by Sun.

THM Rogers-S 2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

- By the first Strehl identity, $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} = \sum_{k=0}^{n} \binom{n}{k}^3.$

Personal encounter in the wild III: Series for $1/\pi$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\text{modular function}} = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau).$$

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (42n + 5) \frac{1}{2^{6n}} = \frac{16}{\pi}$$

Personal encounter in the wild III: Series for $1/\pi$

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FACT

- For $\tau \in \mathbb{Q}(\sqrt{-d})$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a **quasimodular** form.
- The prototypical $E_2(\tau)$ satisfies

$$\tau^{-2} E_2(-1/\tau) - E_2(\tau) = \frac{6}{\pi i \tau}.$$

- These are the main ingredients for series for $1/\pi$. Mix and stir.

EXAMPLE IV

Positivity of rational functions

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}$$

Personal encounter in the wild IV: Positivity

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1,\ldots,n_d \geqslant 0} a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1,\ldots,n_d} > 0$ for all indices.

EG The Askey–Gasper rational function $A(x, y, z)$ and the Szegő rational function $S(x, y, z)$ are positive.

$$A(x, y, z) = \frac{1}{1 - (x + y + z) + 4xyz}$$

$$S(x, y, z) = \frac{1}{1 - (x + y + z) + \frac{3}{4}(xy + yz + zx)}$$

- Both functions are on the boundary of positivity.

Personal encounter in the wild IV: Positivity

- WZ shows that the diagonal terms a_n of $A(x,y,z)$ satisfy

$$(n+1)^2 a_{n+1} = (7n^2+7n+2)a_n + 8n^2 a_{n-1}.$$

  This proves that they equal the **Franel numbers**

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^3.$$

- Using the modular parametrization of the associated Calabi–Yau differential equation, we have

$$\sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}\, {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix}\,\middle|\, \frac{27z^2}{(1-2z)^3}\right).$$

- The Kauers–Zeilberger rational function

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}$$

is conjectured to be positive.

- Its positivity implies the positivity of the Askey–Gasper function

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

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The Kauers–Zeilberger function has diagonal coefficients

$$d_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2.$$

Personal encounter in the wild IV: Positivity

- Under what condition(s) is the positivity of a rational function

$$h(x_1, \ldots, x_d) = \frac{1}{\sum_{k=0}^{d} c_k e_k(x_1, \ldots, x_d)}$$

  implied by the positivity of its diagonal?
- Is the positivity of $h(x_1, \ldots, x_{d-1}, 0)$ a sufficient condition?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

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$$h(x, y) = \frac{1}{1 + c_1(x + y) + c_2 xy}$$

is positive iff $h(x, 0)$ and the diagonal of $h(x, y)$ are positive.

OUTLOOK

Very recent results on Apéry numbers

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

Apéry numbers as diagonals

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

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The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

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THM S 2013 The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

- Previously known: they are also the diagonal of

$$\frac{1}{(1 - x_1)\left[(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5) - x_1 x_2 x_3\right]}.$$

- Such identities are routine to proof, but much harder to discover.

- Denote with $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ the coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

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Let $\boldsymbol{n} = (n_1, n_2, n_3, n_4) \in \mathbb{Z}_{\geqslant 0}^4$. For primes $p \geqslant 5$,

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \mod p^{3r}.$$

Multivariable supercongruences

- Denote with $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ the coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

THM
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Let $\boldsymbol{n} = (n_1, n_2, n_3, n_4) \in \mathbb{Z}_{\geqslant 0}^4$. For primes $p \geqslant 5$,

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \mod p^{3r}.$$

- Note that if $$\sum_{n \geqslant 0} a(n) x^n = F(x),$$ $$\zeta_p = e^{2\pi i/p}$$

  then $$\sum_{n \geqslant 0} a(pn) x^{pn} = \frac{1}{p} \sum_{k=0}^{p-1} F(\zeta_p^k x).$$

Just some of the many open problems

- Supercongruences for all Apéry-like numbers
  - proof for all of them
  - uniform explanation
  - multivariable extensions
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- Many further questions remain.
  - is the known list complete?
  - higher-order analogs, Calabi–Yau DEs
  - reason for modularity
  - q-analogs
  - ...

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

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