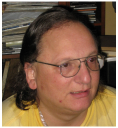
Symbolic evaluation of log-sine integrals in polylogarithmic terms

Joint Mathematics Meetings, Boston, MA

Armin Straub

January 7, 2012

Tulane University, New Orleans



Joint work with: Jon Borwein
U. of Newcastle, AU

Definition

- (Generalized) **log-sine integrals**:

$$\mathrm{Ls}_n^{(k)}(\sigma) = -\int_0^\sigma \theta^k \log^{n-1-k}\left|2\sin\frac{\theta}{2}\right| \mathrm{d}\theta$$

$$\mathrm{Ls}_n(\sigma) = \mathrm{Ls}_n^{(0)}(\sigma)$$

L. Lewin
Polylogarithms and associated functions
North Holland, 1981

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$$\mathrm{Ls}_n(\sigma) = \mathrm{Ls}_n^{(0)}(\sigma)$$

$$\mathrm{Ls}_5(\pi) = -\frac{19}{240}\pi^5$$

$$\mathrm{Ls}_3^{(1)}(\pi) = \frac{7}{4}\zeta(3)$$

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“Queer special numerical results”

$$-\operatorname{Ls}_4^{(1)}\left(\frac{\pi}{3}\right) = \int_0^{\pi/3} \theta \log^2 \left|2 \sin \tfrac{\theta}{2}\right| \, \mathrm{d}\theta = \frac{17}{6480}\pi^4$$

aspects of the trilogarithm in Chapter 6 are, I believe, also novel, as are the log-sine integrals of Chapter 6 and 7. These functions, too, have their queer special numerical results. Perhaps, to keep the torch burning, it would not be out of place to record, for example, the relation

$$\int_0^{\pi/3} \theta \log^2 (2 \sin \tfrac{1}{2}\theta) \, d\theta = 17 \pi^4/6480;$$

maybe in its own way as interesting as any of Landen's curious formulas.

More examples of log-sine integral evaluations

Example (Zucker, 1985)

$$\mathrm{Ls}_6^{(3)}\left(\frac{\pi}{3}\right) - 2\,\mathrm{Ls}_6^{(1)}\left(\frac{\pi}{3}\right) = \frac{313}{204120}\pi^6$$

More examples of log-sine integral evaluations

Example (Zucker, 1985)

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Beware of misprints and human calculators (Lewin, 7.144)

$$\mathrm{Ls}_5^{(2)}(2\pi) = -\frac{13}{45}\pi^5$$
$$\neq \frac{7}{30}\pi^5$$

- There are many more errors/typos in the literature.
- Automated **simplification, validation and correction** tools are more and more important.

Basic log-sine integrals at π

An exponential generating function (Lewin)

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\mathrm{Ls}_{n+1}\left(\pi\right)\frac{\lambda^n}{n!}=\frac{\Gamma\left(1+\lambda\right)}{\Gamma^2\left(1+\frac{\lambda}{2}\right)}=\binom{\lambda}{\frac{\lambda}{2}}$$

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Example (Mathematica)

```
FullSimplify[D[-Pi Binomial[x,x/2], {x,5}] /.x->0]
```

$$\mathrm{Ls}_6(\pi)=\frac{45}{2}\pi\,\zeta(5)+\frac{5}{4}\pi^3\zeta(3)$$

Multiple polylogarithms

- Multiple polylogarithm:

$$\mathrm{Li}_{a_1,\ldots,a_k}(z) = \sum_{n_1>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{a_1}\cdots n_k^{a_k}}$$

- Multiple zeta values:

$$\zeta(a_1,\ldots,a_k) = \mathrm{Li}_{a_1,\ldots,a_k}(1)$$

Log-sine integrals at π

Theorem (Lewin, Borwein-S)

$$-\sum_{n,k\geqslant 0} \mathrm{Ls}_{n+k+1}^{(k)}(\pi)\,\frac{\lambda^n}{n!}\frac{(i\mu)^k}{k!} = i\sum_{n\geqslant 0}\binom{\lambda}{n}\frac{(-1)^n \mathrm{e}^{i\pi\frac{\lambda}{2}} - \mathrm{e}^{i\pi\mu}}{n+\mu-\frac{\lambda}{2}}$$

Log-sine integrals at π

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Example

$$-\mathrm{Ls}_4^{(2)}(\pi) = \frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\frac{\mathrm{d}}{\mathrm{d}\lambda} i\sum_{n\geqslant 0}\binom{\lambda}{n}\frac{(-1)^n \mathrm{e}^{i\pi\frac{\lambda}{2}} - \mathrm{e}^{i\pi\mu}}{\mu-\frac{\lambda}{2}+n}\Bigg|_{\lambda=\mu=0}$$

$$= 2\pi\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^3} = \frac{3}{2}\pi\zeta(3)$$

Log-sine integrals at π

Theorem (Lewin, Borwein-S)

$$-\sum_{n,k\geqslant 0} \mathrm{Ls}^{(k)}_{n+k+1}(\pi) \frac{\lambda^n}{n!} \frac{(i\mu)^k}{k!} = i \sum_{n\geqslant 0} \binom{\lambda}{n} \frac{(-1)^n \mathrm{e}^{i\pi\frac{\lambda}{2}} - \mathrm{e}^{i\pi\mu}}{n+\mu-\frac{\lambda}{2}}$$

$$\frac{(-1)^\alpha}{\alpha!} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^\alpha \binom{\lambda}{n}\bigg|_{\lambda=0} = \frac{(-1)^n}{n} \underbrace{\sum_{n>i_1>i_2>\ldots>i_{\alpha-1}} \frac{1}{i_1 i_2 \cdots i_{\alpha-1}}}_{H^{[\alpha-1]}_{n-1}}$$

Note: $\displaystyle\sum_{n\geqslant 1} \frac{(\pm 1)^n}{n^\beta} H^{[\alpha]}_{n-1} = \mathrm{Li}_{\beta,\{1\}^\alpha}(\pm 1)$

Thus $\mathrm{Ls}^{(k)}_n(\pi)$ evaluates in terms of the Nielsen polylogs $\mathrm{Li}_{\beta,\{1\}^\alpha}(\pm 1)$.

Log-sine integrals at general arguments

In general, $\mathrm{Ls}_n^{(k)}(\tau)$ evaluates in terms of Nielsen polylogs $\mathrm{Li}_{\beta,\{1\}^\alpha}(\mathrm{e}^{i\tau})$.

$$\mathrm{Ls}_4^{(1)}(\tau) = \frac{1}{180}\pi^4 - 2\,\mathrm{Gl}_{3,1}(\tau) - 2\tau\,\mathrm{Gl}_{2,1}(\tau) - \frac{1}{16}\tau^4 + \frac{1}{6}\pi\tau^3 - \frac{1}{8}\pi^2\tau^2$$

- $\mathrm{Gl}_{2,1}(\tau) = \mathrm{Im}\ \mathrm{Li}_{2,1}(e^{i\tau})$
- $\mathrm{Gl}_{3,1}(\tau) = \mathrm{Re}\ \mathrm{Li}_{3,1}(e^{i\tau})$

Implementation

```
In[1]:= << "~/docs/math/mathematica/logsine.m"

LsToLi: evaluating log-sine integrals in polylogarithmic terms
     accompanying the paper "Special values of generalized log-sine integrals"
     -- Jonathan M. Borwein, University of Newcastle
     -- Armin Straub, Tulane University
     -- Version 1.3 (2011/03/11)
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In[2]:= Ls[4, 1, Pi / 3] // LsToLi
```

Out[2]= $-\frac{17\,\pi^4}{6480}$

Implementation

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```

In[2]:= `Ls[4, 1, Pi / 3] // LsToLi`

Out[2]= $-\frac{17\,\pi^4}{6480}$

In[3]:= `$Assumptions = 0 < τ < Pi;`
`Ls[4, 1, τ] // LsToLi`

Out[4]= $\frac{\pi^4}{180} - \frac{\pi^2\,\tau^2}{8} + \frac{\pi\,\tau^3}{6} - \frac{\tau^4}{16} - 2\,\tau\,\mathrm{Gl}[\{2, 1\}, \tau] - 2\,\mathrm{Gl}[\{3, 1\}, \tau]$

Numerical usage

Just using Mathematica 7, a few hundred digits are easy:

```
ls52 = Ls[5, 2, 2 Pi / 3] // LsToLi
```

$$-\frac{8\pi^5}{1215} - \frac{8}{9}\pi^2\,\mathrm{Gl}\left[\{2, 1\}, \frac{2\pi}{3}\right] - \frac{8}{3}\pi\,\mathrm{Gl}\left[\{3, 1\}, \frac{2\pi}{3}\right] + 4\,\mathrm{Gl}\left[\{4, 1\}, \frac{2\pi}{3}\right]$$

```
N[ls52]
```

```
-0.518109
```

```
N[ls52, 200]
```

```
-0.518108786829680117347265638731696755021879668243153214067389472482464930592067915068175917962342634092283168874070625727137897015228328288301238053344434601555482416349687142642605456956152340876808799
```


M. Kalmykov and A. Sheplyakov
lsjk - a C++ library for arbitrary-precision numeric evaluation of the generalized log-sine functions
Comput. Phys. Commun., 172(1):45–59, 2005

Numerical usage

With specialized polylog routines, a few thousand digits are easy:

```
N[ls52 /. Gl → GlN, 1500]

-0.5181087868296801173472656387316967550218796682431532140673894724824649305920679150681\
  75917962342634092283168874070625727137897015228328288301238053344434601555482416349687\
  14264260545695615234087680878833012527445245320056506539166335466076425659394333250236\
  8704996964072618430077108019449120638389717243843114495652058348073506174420066399192 0\
  9369665418959139680545328024232441688700377283742072772714029093211428066255503314839 3\
  4346157017999680014851656153847980079446225103428769237020891516271578761307434220860 7\
  9568427226785119431613041566002524816863193071905942805038395349632095408925090936865 0\
  7634802118402367023958364480605724883286048232501257730562643059641954715004420327604 8\
  0010663468680842589408031170729595597489331212416860550548109691664311874770739977269 9\
  5685152764342004787909957595709564369273416476440874928297302997322623212162505556681 8\
  3011472959999435674673619440973336383204370234472148571069312485138137784268429835976 2\
  7902662692522141735091653899270134033573580034488504210827748452303627506494601119042 4\
  5981100513011527807257203511249099602357226702813690561684728688359473915676500087227 4\
  1171715732532324304909392481203408934403135504697147308473633542555204308358155331186 4\
  6643809518680164612459364379017456732824209636267198665451044904639579951516434428642 2\
  0863570350441280524570356600335366814158891280187202197951693897999883945925408859044 9\
  5838444532048484183137311323287438168230950278471856582525113193260785397060334327453 1\
  023502381511423660445880027683673379705
```

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(Multiple) Mahler measure

$$\mu(p_1, \ldots, p_k) := \int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} \log \left| p_i \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n} \right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

An example considered by Sasaki put in log-sine form

$$\begin{aligned} \mu(1+x+y_1, 1+x+y_2, \ldots, 1+x+y_k) &= \int_{1/6}^{5/6} \log^k \left| 1 - e^{2\pi i\, t} \right| \mathrm{d}t \\ &= \frac{1}{\pi} \mathrm{Ls}_{k+1}\left(\frac{\pi}{3} \right) - \frac{1}{\pi} \mathrm{Ls}_{k+1}(\pi) \end{aligned}$$

(Multiple) Mahler measure

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$$\mu(1 + x + y_1, 1+x + y_2, \ldots, 1 + x + y_k)$$
$$= \int_{1/6}^{5/6} \log^k \left| 1 - e^{2\pi i\, t} \right| \mathrm{d}t$$
$$= \frac{1}{\pi} \mathrm{Ls}_{k+1} \left(\frac{\pi}{3} \right) - \frac{1}{\pi} \mathrm{Ls}_{k+1} (\pi)$$

Example (Kurokawa-Lalín-Ochiai)

$$\mu(1 + x + y_1, 1 + x + y_2) = \frac{\pi^2}{54}$$

Inverse binomial sums

A very classical example

$$\sum_{j=1} \frac{1}{j^4} \frac{1}{\binom{2j}{j}} = \frac{17}{36}\zeta(4)$$

Inverse binomial sums

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$$\sum_{j=1} \frac{1}{j^4} \frac{1}{\binom{2j}{j}} = \frac{17}{36}\zeta(4)$$

This is a special case of the general relation:

Theorem (Borwein-Broadhurst-Kamnitzer)

$$\sum_{j=1}^{\infty} \frac{1}{j^{n+2}} \frac{1}{\binom{2j}{j}} = -\frac{(-2)^n}{n!} \operatorname{Ls}_{n+2}^{(1)}\left(\frac{\pi}{3}\right)$$

J. M. Borwein, D. J. Broadhurst, and J. Kamnitzer.
Central binomial sums, multiple Clausen values, and zeta values.
Experimental Mathematics, 10(1):25–34, 2001.

Inverse binomial sums—scratching an iceberg

$u = 4\sin^2\frac{\theta}{2}$ (so $u = 1$ if $\theta = \frac{\pi}{3}$)
$l_\theta = \log\left(2\sin\frac{\theta}{2}\right)$, $L_\theta = \log\left(2\cos\frac{\theta}{2}\right)$

The connection is deeper:

Example (Davydychev-Kalmykov)

$$\sum_{j=1}^{\infty} \frac{u^j}{j^{n+2}} \frac{1}{\binom{2j}{j}} = -\sum_{m=0}^{n} \frac{(-2)^m (2l_\theta)^{n-m}}{m!(n-m)!} \mathrm{Ls}_{m+2}^{(1)}(\theta)$$

By the way:
$u = -1$ if $\theta = 2i\log\frac{1+\sqrt{5}}{2}$

A. Davydychev and M. Kalmykov.
Massive Feynman diagrams and inverse binomial sums.
Nuclear Physics B, 699(1-2):3–64, 2004.

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$$\sum_{j=1}^{\infty} \frac{u^j}{j^{n+2}} \frac{1}{\binom{2j}{j}} S_2(j-1) = -\frac{1}{6} \sum_{m=0}^{n} \frac{(-2)^m (2l_\theta)^{n-m}}{m!(n-m)!} \operatorname{Ls}_{m+4}^{(3)}(\theta)$$

- Here: $S_a(j) = \sum_{i=1}^{j} \frac{1}{j^a}$

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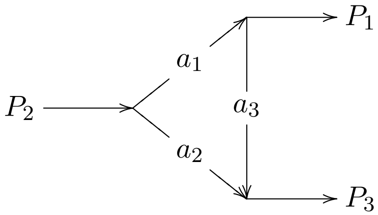
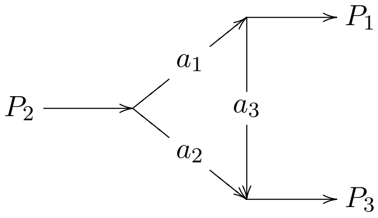
Feynman diagrams

- 

Propagator associated to the index a_1 has mass m

$P_1^2 = P_3^2 = 0$ and
$P_2^2 = (P_1 + P_3)^2 = s$

Feynman diagrams

- 

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- This **Feynman diagram** evaluates as:

$$\int \frac{\mathrm{d}^D q}{i\pi^{D/2}} \frac{1}{[(P_1+q)^2 - m^2]^{a_1} \, [(P_3-q)^2]^{a_2} \, [q^2]^{a_3}}$$

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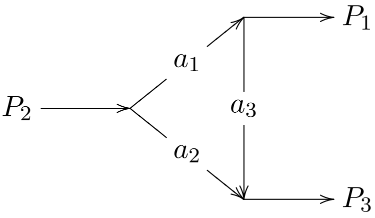
- 

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- In the special case $a_1 = a_2 = a_3 = 1$ and $D = 4 - 2\epsilon$ this becomes:

$$-(m^2)^{-1-\epsilon}\Gamma(\epsilon - 1)\, {}_2F_1\left(\begin{matrix} 1+\epsilon, 1 \\ 2-\epsilon \end{matrix} \middle| \frac{s}{m^2}\right)$$

ε-expansion of a hypergeometric function

- $[\varepsilon^n]\ {}_3F_2\left(\left.\begin{array}{c}\frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2}\end{array}\right| \frac{1}{4}\right)$

ε-expansion of a hypergeometric function

- $[\varepsilon^n]\ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2}\end{matrix} \middle| \frac{1}{4}\right) = (-1)^n \sum_{j=1}^{\infty} \frac{2}{j} \frac{1}{\binom{2j}{j}} \sum \prod_{k=1}^{n} \frac{A_{k,j}^{m_k}}{m_k! k^{m_k}}$
  where the inner sum is over all $m_1, \ldots, m_n \geqslant 0$ such that $m_1 + 2m_2 + \ldots + nm_n = n$, and $A_{k,j} = S_k(2j-1) - 2^{2-k} S_k(j-1) - 1$

ε-expansion of a hypergeometric function

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  where the inner sum is over all $m_1, \ldots, m_n \geqslant 0$ such that $m_1 + 2m_2 + \ldots + nm_n = n$, and $A_{k,j} = S_k(2j-1) - 2^{2-k} S_k(j-1) - 1$

Example

$$[\varepsilon]\ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}\\ 1, \frac{\varepsilon+3}{2}\end{matrix}\middle|\frac{1}{4}\right) = -\sum_{j=1}^{\infty} \frac{2}{j} \frac{1}{\binom{2j}{j}} \left[S_1(2j-1) - 2S_1(j-1) - 1\right]$$
$$= \frac{2}{3\sqrt{3}} \left[\pi - \pi \log 3 + \mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]$$

Moments of random walks

- We were interested in the ε-expansion of

$$ {}_3F_2\left(\begin{matrix} \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2} \end{matrix} \middle| \frac{1}{4}\right) $$

for a different reason: they came up in our study of random walks

D. Borwein, J. M. Borwein, A. Straub, and J. Wan.
Log-sine evaluations of Mahler measures, II.
Preprint, 2011.
arXiv:1103.3035.

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  for a different reason: they came up in our study of random walks
- The terms determine the **higher Mahler measures**:

$$ \mu_k(1+x+y) = \mu(\underbrace{1+x+y, 1+x+y, \ldots}_{k \text{ many}}) $$

D. Borwein, J. M. Borwein, A. Straub, and J. Wan.
Log-sine evaluations of Mahler measures, II.
Preprint, 2011.
arXiv:1103.3035.

THANK YOU!

- Slides for this talk will be available from my website:
  `http://arminstraub.com/talks`
- The implementation of our results is also freely available at:
  `http://arminstraub.com/pub/log-sine-integrals`

J. M. Borwein and A. Straub.
Special values of generalized log-sine integrals.
Proceedings of ISSAC 2011 (International Symposium on Symbolic and Algebraic Computation), 2011.
arXiv:1103.4298.

J. M. Borwein and A. Straub.
Log-sine evaluations of Mahler measures.
J. Aust Math. Soc., 2011.
arXiv:1103.3893.

...

Reductions of polylogarithms

- The (Nielsen) polylogarithms satisfy various relations:

Example (Weight 4, $2\sigma = \tau - \pi$)

$$\mathrm{Cl}_{3,1}(\tau) = \mathrm{Cl}_4(\tau) - \sigma\,\mathrm{Cl}_3(\tau) + \sigma\zeta(3)$$

$$\mathrm{Gl}_4(\tau) = -\frac{\sigma^4}{3} + \zeta(2)\sigma^2 - \frac{7\zeta(4)}{8}$$

$$\mathrm{Gl}_{2,1,1}(\tau) = \frac{1}{2}\mathrm{Gl}_{3,1}(\tau) + \sigma\,\mathrm{Gl}_{2,1}(\tau) + \frac{\sigma^4}{6} - \frac{\zeta(4)}{16}$$

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$$\mathrm{Cl}_{3,1}(\tau) = \mathrm{Cl}_4(\tau) - \sigma\,\mathrm{Cl}_3(\tau) + \sigma\zeta(3)$$

$$\mathrm{Gl}_4(\tau) = -\frac{\sigma^4}{3} + \zeta(2)\sigma^2 - \frac{7\zeta(4)}{8}$$

$$\mathrm{Gl}_{2,1,1}(\tau) = \frac{1}{2}\,\mathrm{Gl}_{3,1}(\tau) + \sigma\,\mathrm{Gl}_{2,1}(\tau) + \frac{\sigma^4}{6} - \frac{\zeta(4)}{16}$$

- For the special argument $\tau = \pi/3$ there are additional relations:

$$\mathrm{Cl}_{2,1,1}\left(\frac{\pi}{3}\right) = \mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{\pi^2}{18}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi}{9}\zeta(3)$$

Log-sine integrals at $\pi/3$

- $\mathrm{Ls}_n^{(k)}\left(\frac{\pi}{3}\right)$ evaluates in terms of polylogarithms at $\omega = \mathrm{e}^{i\pi/3}$ — the sixth root of unity
- Crucial: $1 - \omega = \overline{\omega} = \omega^2$

$$\mathrm{Li}_{2,1,1}(x) = \frac{\pi^4}{90} - \frac{1}{6}\log(1-x)^3 \log x - \frac{1}{2}\log(1-x)^2 \,\mathrm{Li}_2(1-x) + \log(1-x)\,\mathrm{Li}_3(1-x) - \mathrm{Li}_4(1-x)$$

- Currently, our implementation has a table of extra reductions built in — a systematic study is under way