# $q$-binomial coefficient congruences

**CARMA Analysis and Number Theory Seminar**

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# Our first $q$-analogs

- The natural number $n$ has the $q$-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \dots q^{n-1}$$

In the limit $q \to 1$ a $q$-analog reduces to the classical object.

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In the limit $q \to 1$ a $q$-analog reduces to the classical object.

- The $q$-factorial:

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

- The $q$-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \binom{n}{n-k}_q$$

D1

# A $q$-binomial coefficient

**Example**

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$
$$= (1-q+q^2)\underbrace{(1+q+q^2)}_{=[3]_q}\underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

- Let us understand the first term a bit better!

# Cyclotomic polynomials

The $n$th cyclotomic polynomial:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k < n \\ (k,n)=1}} (q - \zeta^k) \qquad \text{where } \zeta = e^{2\pi i/n}$$

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- $[n]_q = \dfrac{q^n - 1}{q - 1} = \prod_{\substack{1 < d \leqslant n \\ d \mid n}} \Phi_d(q)$ For primes: $[p]_q = \Phi_p(q)$
- An irreducible monic integral polynomial is cyclotomic if and only if its Mahler measure $\prod \max(|\alpha|, 1)$ is $1$.

# Some cyclotomic polynomials exhibited

- $\Phi_{mn}(q) = \Phi_n(q^m)$ if $m|n$
- $\Phi_{2n}(q) = \Phi_n(-q)$ for odd $n > 1$
- $\Phi_n(q)$ is palindromic

**Example**

$$\Phi_2(q) = q + 1$$

$$\Phi_3(q) = q^2 + q + 1$$

$$\Phi_6(q) = q^2 - q + 1$$

$$\Phi_9(q) = q^6 + q^3 + 1$$

$$\Phi_{21}(q) = q^{12} - q^{11} + q^9 - q^8 + q^6 - q^4 + q^3 - q + 1$$

$$\vdots$$

$$\Phi_{102}(q) = q^{32} + q^{31} - q^{29} - q^{28} + q^{26} + q^{25} - q^{23} - q^{22} + q^{20} + q^{19} - q^{17} - q^{16} - q^{15} + q^{13} + q^{12} - q^{10} - q^9 + q^7 + q^6 - q^4 - q^3 + q + 1$$

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# Back to $q$-binomials

- $[n]_q = \dfrac{q^n - 1}{q - 1} = \prod_{\substack{1 < d \leqslant n \\ d | n}} \Phi_d(q)$
- $\binom{n}{k}_q = \dfrac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q [k-1]_q \cdots [1]_q}$
- How often does $\Phi_d(q)$ appear in this?
  - It appears $\left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-k}{d} \right\rfloor - \left\lfloor \frac{k}{d} \right\rfloor$ times

# Back to $q$-binomials

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  - Obviously nonnegative: the $q$-binomials are indeed **polynomials**
  - Also at most one: **square-free**
  - $\dbinom{n}{k}_q$ always contains $\Phi_n(q)$ if $0 < k < n$.
- Good way to compute $q$-binomials

  and even get them factorized for free

# The coefficients of $q$-binomial coefficients

- Here's some $q$-binomials in **expanded** form:

**Example**

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\binom{9}{3}_q = q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1$$

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**What is the degree of the $q$-binomial?** It is $(n-k)k$.

- All coefficients are positive!
- In fact, the coefficients are **unimodal**.

Sylvester, 1878

# $q$-binomials: Pascal's triangle

Define the $q$-binomials via the $q$-Pascal rule:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$$

D2

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$$\binom{4}{2}_q = 1+q+q^2+q^2(1+q+q^2) = 1+q+2q^2+q^3+q^4$$

$$\binom{n}{k}_q = \sum_{S \in \binom{n}{k}} q^{w(S)} \quad \text{where } w(S) = \sum_j s_j - j$$

D3

$w(S) =$ "normalized sum of $S$"

**Example**

$$\underbrace{\{1,2\}}_{\to 0}, \underbrace{\{1,3\}}_{\to 1}, \underbrace{\{1,4\}}_{\to 2}, \underbrace{\{2,3\}}_{\to 2}, \underbrace{\{2,4\}}_{\to 3}, \underbrace{\{3,4\}}_{\to 4}$$

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- $k$-element subsets of $n$ whose normalized sum is $m$
- words made from $k$ ones and $n-k$ twos which have $m$ inversions
- partitions $\lambda$ of $m$ whose Ferrer's diagram fits in a $k \times (n-k)$ box

# $q$-Chu-Vandermonde

Different representations make different properties apparent!

- Chu-Vandermonde: $$\binom{m+n}{k} = \sum_j \binom{m}{j}\binom{n}{k-j}$$

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# Automatic proving of $q$-identities

In[1]:=
```
<< "~/docs/math/mathematica/packages/qZeil.m";
```

```
q-Zeilberger Package by Axel Riese — © RISC Linz — V 2.42 (02/18/05)
```

In[2]:=
```
qZeil[qBinomial[m, j, q] qBinomial[n, k - j, q] q^((m - j) (k - j)), {j, 0, m + n}, k, 1]
```

Out[2]=
$$\mathrm{SUM}[k] == \frac{\left(1 - q^{1-k+m+n}\right)\,\mathrm{SUM}[-1+k]}{1 - q^k}$$


**P. Paule and A. Riese**
*A Mathematica q-Analogue of Zeilberger's Algorithm Based on an Algebraically Motivated Approach to q-Hypergeometric Telescoping*
Fields Inst. Commun., Vol. 14, 1997

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Out[2]= $\mathrm{SUM}[k] == \dfrac{\left(1 - q^{1-k+m+n}\right)\,\mathrm{SUM}[-1+k]}{1 - q^{k}}$

- **Encoded** implementation in Mathematica at risk of **bit rot**?

  last version of qZeil by Alex Riese from 2005 — many examples don't work in MMA7 anymore. . .

- Sage as a solution?


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Let $q$ be a prime power.

$$\binom{n}{k}_q = \text{number of } k\text{-dim. subspaces of } \mathbb{F}_q^n$$

D4

# $q$-binomials: algebraic

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D4

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$$(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$$

- Hence the number of $k$-dim. subspaces of $\mathbb{F}_q^n$ is:

$$\frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \binom{n}{k}_q$$

# $q$-binomials: noncommuting variables

Suppose $yx = qxy$ where $q$ commutes with $x, y$. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}$$

D5

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**Example**

$$\binom{4}{2}_q x^2 y^2 = xxyy + xyxy + xyyx + yxxy + yxyx + yyxx$$
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- Let $X \cdot f(x) = xf(x)$ and $Q \cdot f(x) = f(qx)$. Then:

$$QX \cdot f(x) = qxf(qx) = qXQ \cdot f(x)$$

# $q$-calculus

It all starts with the $q$-**derivative**:

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- Define $e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$

- $D_q e_q^x = e_q^x$
- $e_q^x \cdot e_q^y \neq e_q^{x+y}$
  unless $yx = qxy$
- $e_q^x \cdot e_{1/q}^{-x} = 1$

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- Define $e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$
- **Homework:** Define $\cos_q(x)$, $\sin_q(x)$, ... and develop some $q$-trigonometry.

- $D_q e_q^x = e_q^x$
- $e_q^x \cdot e_q^y \neq e_q^{x+y}$ unless $yx = qxy$
- $e_q^x \cdot e_{1/q}^{-x} = 1$

# $q$-calculus: the $q$-integral

- Formally inverting $D_q F(x) = f(x)$ gives:

$$F(x) = \int_0^x f(x) d_q x := (1-q) \sum_{n=0}^{\infty} q^n x f(q^n x)$$

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**Theorem (Fundamental theorem of $q$-calculus)**

*Let $0 < q < 1$. Then*

$$D_q F(x) = f(x).$$

*$F(x)$ is the unique such function continuous at $0$ with $F(0) = 0$.*

*Fineprint: one needs for instance that $|f(x) x^{\alpha}|$ is bounded on some $(0, a]$.*

# $q$-calculus: special functions

- Define the $q$-gamma function as

$$\Gamma_q(s) = \int_0^\infty x^{s-1} e_{1/q}^{-qx} d_q x$$

- $\Gamma_q(s+1) = [s]_q \, \Gamma_q(s)$
- $\Gamma_q(n+1) = [n]_q!$

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D6

$q$-beta function:

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x$$

- $B_q(t,s) = \dfrac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}$
- $B_q(t,s) = B_q(s,t)$

- Here, $(x-a)_q^n$ is defined by:

$$f(x) = \sum_{n \geqslant 0} (D_q^n f)(a) \frac{(x-a)_q^n}{[n]_q!}$$

Explicitly: $(x-a)_q^n = (x-a)(x-qa)\cdots(x-q^{n-1}a)$

# Summary: the $q$-binomial coefficient

- The $q$-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

- Via a $q$-version of Pascal's rule
- Combinatorially, as the generating function of the element sums of $k$-subsets of an $n$-set
- Algebraically, as the number of $k$-dimensional subspaces of $\mathbb{F}_q^n$
- Via a binomial theorem for noncommuting variables
- Analytically, via $q$-integral representations
- We have not touched: quantum groups arising in representation theory and physics

# Classical binomial congruences

John Wilson (1773, Lagrange): $(p-1)! \equiv -1 \mod p$

Charles Babbage (1819): $\binom{2p-1}{p-1} \equiv 1 \mod p^2$

Joseph Wolstenholme (1862): $\binom{2p-1}{p-1} \equiv 1 \mod p^3$

James W.L. Glaisher (1900): $\binom{mp-1}{p-1} \equiv 1 \mod p^3$

Wilhelm Ljunggren (1952): $\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$

# Wilson's congruence

**Wilson's congruence (Lagrange, 1773)**

$$(p-1)! \equiv -1 \mod p$$

- known to Ibn al-Haytham, ca. 1000 AD
- This congruence holds **if and only if** $p$ is a prime.
- Not great as a practical primality test though...

> *The problem of distinguishing prime numbers from composite numbers ... is known to be one of the most important and useful in arithmetic. ... The dignity of the science itself seems to require that every possible means be explored for the solution of a problem so elegant and so celebrated.*
>
> **C. F. Gauss**, *Disquisitiones Arithmeticae, 1801*

# Babbage's congruence

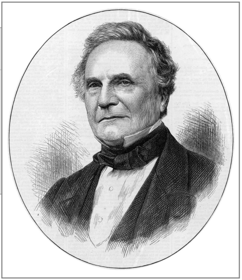
$(n-1)!+1$ is divisible by $n$ if and only if $n$ is a prime number

> *In attempting to discover some analogous expression which should be divisible by $n^2$, whenever $n$ is a prime, but not divisible if $n$ is a composite number . . .* Charles Babbage is led to:

**Theorem (Babbage, 1819)**

*For primes $p \geqslant 3$:*

$$\binom{2p-1}{p-1} \equiv 1 \mod p^2$$



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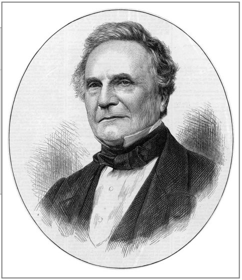


- $$\binom{2n-1}{n-1} = \frac{(n+1)(n+2)\cdots(2n-1)}{1 \cdot 2 \cdots (n-1)}$$

# Babbage's congruence

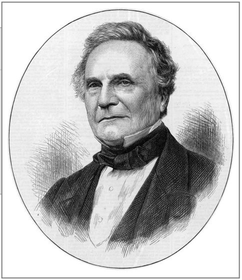
$(n-1)! + 1$ is divisible by $n$ if and only if $n$ is a prime number

> *In attempting to discover some analogous expression which should be divisible by $n^2$, whenever $n$ is a prime, but not divisible if $n$ is a composite number …* Charles Babbage is led to:

**Theorem (Babbage, 1819)**

*For primes $p \geqslant 3$:*

$$\binom{2p-1}{p-1} \equiv 1 \mod p^2$$



- $\binom{2n-1}{n-1} = \frac{(n+1)(n+2)\cdots(2n-1)}{1 \cdot 2 \cdots (n-1)}$
- Does not quite characterize primes!

$n = 16843^2$

# A simple combinatorial proof

- We have

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k}$$

- Note that $p$ divides $\binom{p}{k}$ unless $k = 0$ or $k = p$.

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- Note that $p$ divides $\binom{p}{k}$ unless $k = 0$ or $k = p$.
- $\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p}$ which is only trouble when $p = 2$

# A $q$-analog of Babbage's congruence

- Using $q$-Chu-Vandermonde

$$\binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q q^{(p-k)^2}$$
$$\equiv q^{p^2} + 1 \qquad \mod [p]_q^2$$

- Again, $[p]_q$ divides $\binom{p}{k}_q$ unless $k = 0$ or $k = p$.

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**Theorem**

$$\binom{2p}{p}_q \equiv [2]_{q^{p^2}} \mod [p]_q^2$$

# Extending the $q$-analog

- Actually, the same argument shows:

**Theorem (W. Edwin Clark, 1995)**

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- Sketch of the corresponding classical congruence:

$$\binom{ap}{bp} = \sum_{k_1+\ldots+k_a=bp} \binom{p}{k_1}\cdots\binom{p}{k_a}$$
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Similar results by Andrews; e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \mod [p]_q^2$$

**George Andrews**
*$q$-analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher*
Discrete Mathematics 204, 1999

$p^2$

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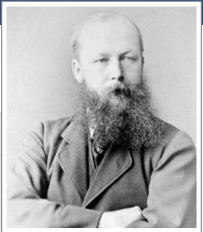
# Wolstenholme and Ljunggren



- Amazingly, the congruences hold modulo $p^3$!

**Theorem (Wolstenholme, 1862)**

*For primes $p \geqslant 5$:* $$\binom{2p-1}{p-1} \equiv 1 \mod p^3$$

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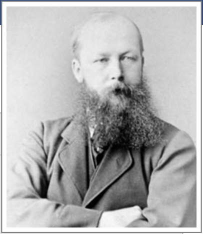

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**Theorem (Ljunggren, 1952)**

*For primes* $p \geqslant 5$*:*
$$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$$

- Note the restriction on $p$ — proofs are **algebraic**.

# Proof of Wolstenholme's congruence

$$\binom{2p-1}{p-1} = \frac{(2p-1)(2p-2)\cdots(p+1)}{1\cdot 2\cdots(p-1)}$$
$$= (-1)^{p-1}\prod_{k=1}^{p-1}\left(1-\frac{2p}{k}\right)$$

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$$= \prod_{k=1}^{p-1}\left(1+\frac{p}{k}\right)$$

$$= (-1)^{p-1}\prod_{k=1}^{p-1}\left(1-\frac{2p}{k}\right)$$

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- Wolstenholme's congruence therefore follows from the fractional congruences

$$\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \mod p^2, \tag{1}$$

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- Comparing the $p^3$ residues on the previous slide now shows (1).

# Congruences for $q$-harmonic numbers

**Theorem (Shi-Pan, 2007)**

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q} \equiv -\frac{p-1}{2}(q-1) + \frac{p^2-1}{24}(q-1)^2[p]_q \mod [p]_q^2$$

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**Example ($p = 5$)**

$$\sum_{i=1}^{4} \frac{1}{[i]_q^2} = \frac{\left(q^4+q^3+q^2+q+1\right)\left(q^6+3q^5+7q^4+9q^3+11q^2+6q+4\right)}{(q+1)^2\left(q^2+1\right)^2\left(q^2+q+1\right)^2}$$

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- Equivalent congruences can be given for $\displaystyle\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^n}$

  This choice actually appears a bit more natural

# An exemplatory proof

- We wish to prove

$$\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^2} \equiv -\frac{p^2-1}{12}(1-q)^2 \mod [p]_q$$

**Ling-Ling Shi and Hao Pan**
*A q-Analogue of Wolstenholme's Harmonic Series Congruence*
The American Mathematical Monthly, 144(6), 2007

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- Hence we need to prove: $G(\zeta^m) = -\frac{p^2-1}{12}$ for $m = 1, 2, \ldots, p-1$

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- $G(\zeta^m) = \sum_{i=1}^{p-1} \frac{\zeta^{mj}}{(1-\zeta^{mj})^2} = \sum_{i=1}^{p-1} \frac{\zeta^j}{(1-\zeta^j)^2} = G(\zeta)$

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# An exemplatory proof II

- Define $G(q,z) = \sum_{i=1}^{p-1} \frac{q^i}{(1-q^i z)^2}$
- We need $G(\zeta, 1) = -\frac{p^2-1}{12}$

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$$G(\zeta,z)=\sum_{i=1}^{p-1}\zeta^i\sum_{k=0}^{\infty}\zeta^{ki}(k+1)z^k$$

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$$= \sum_{k=1}^{\infty} k z^{k-1} \ldots$$

This is beautifully generalized in:

**Karl Dilcher**
*Determinant expressions for q-harmonic congruences and degenerate Bernoulli numbers*
Electronic Journal of Combinatorics 15, 2008

$$= p \sum_{k=1}^{\infty} pk z^{pk-1} - \sum_{k=1}^{\infty} k z^{k-1}$$

$$= \frac{p^2 z^{p-1}}{(1-z^p)^2} - \frac{1}{(1-z)^2} \xrightarrow{\text{as } z\to 1} -\frac{p^2-1}{12}$$

# A $q$-analog of Ljunggren's congruence

**Theorem (S, 2010)**

*For primes $p \geqslant 5$:*

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**Example**

Choosing $p = 13$, $a = 2$, and $b = 1$, we have

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1 + q + \ldots + q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132}$ is an irreducible polynomial with integer coefficients.

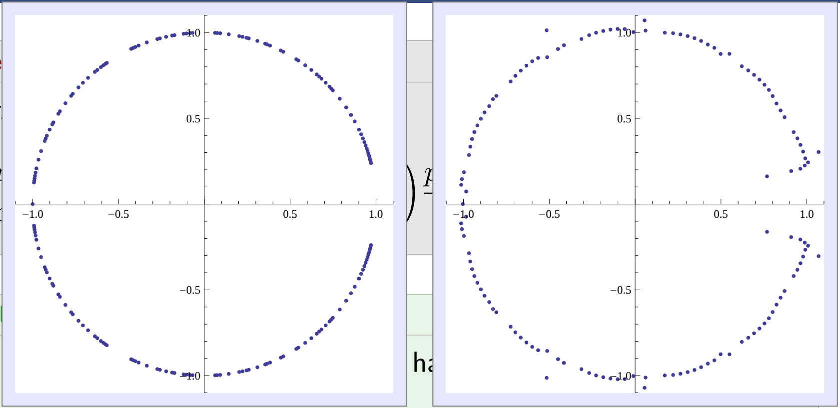
**Theore**

*For pr*

$\binom{a}{b}$ ... $p$

**Exampl**

Choos ... ha

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# Just coincidence?

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

- Ernst Jacobsthal (1952) proved that Ljunggren's classical congruence holds modulo $p^{3+r}$ where $r$ is the $p$-adic valuation of

$$ab(a-b)\binom{a}{b} = 2a\binom{a}{b+1}\binom{b+1}{2}.$$

- It would be interesting to see if this generalization has a nice analog in the $q$-world.

# What happens for composite numbers?

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

**Example ($n = 12$, $a = 2$, $b = 1$)**

$$\binom{24}{12}_q = 1 + q^{144} - \frac{143}{12}(q^{12}-1)^2 + \frac{1}{12}(\underbrace{1 - q^2 + q^4}_{\Phi_{12}(q)})^3 f(q)$$

where $f(q) = 143 + 12q + 453q^2 + \ldots + 12q^{131}$ is an irreducible polynomial with integer coefficients.

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- Ljunggren's $q$-congruence holds modulo $\Phi_n(q)^3$ if $(n, 6) = 1$
  over integer coefficient polynomials! — otherwise we get rational coefficients

# Can we do better than modulo $p^3$?

- Are there primes $p$ such that

$$\binom{2p-1}{p-1} \equiv 1 \mod p^4?$$

- Such primes are called **Wolstenholme primes**.
- The only two known are $16843$ and $2124679$. McIntosh, 1995: up to $10^9$

**C. Helou and G. Terjanian**
*On Wolstenholme's theorem and its converse*
Journal of Number Theory 128, 2008

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- Conjecturally, Wolstenholme's congruence characterizes primes:

$$\binom{2n-1}{n-1} \equiv 1 \mod n^3 \quad \iff \quad n \text{ is prime}$$

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Journal of Number Theory 128, 2008

# Some open problems

- Extension to Jacobsthal's result?
- Extension to

$$\binom{ap}{bp} \equiv \binom{a}{b} \cdot \left[1 - ab(a-b)\frac{p^3}{3} B_{p-3}\right] \mod p^4,$$

and insight into Wolstenholme primes?

- Is there a nice $q$-analog for Gauss' congruence?

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2a \mod p$$

where $p = a^2 + b^2$ and $a \equiv 1$ mod $4$.

Generalized to $p^2$ and $p^3$ by Chowla-Dwork-Evans (1986) and by Cosgrave-Dilcher (2010)

# THANK YOU!

- Slides for this talk will be available from my website:
  http://arminstraub.com/talks

**Victor Kac and Pokman Cheung**
*Quantum Calculus*
Springer, 2002

**Armin Straub**
*A $q$-analog of Ljunggren's binomial congruence*
Proceedings of FPSAC, 2011