Positivity of Szegö's Rational Function

Errata and addenda

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In order to improve on the bound $a \leqslant 0$ in [Str08, Corollary 2] we prove that $h_{a,0}$ is positive only if $a \leqslant 3/4$. The following proof has been kindly suggested by Bruno Salvy. Thank you!

Lemma 1. *The function $h_{a,0}$ defined by*

$$h_{a,0}(x, y, z) = \frac{1}{1 - (x + y + z) + a\,(x\,y + y\,z + z\,x)}$$

is positive only if $a \leqslant 3/4$.

Proof. Now, suppose $a > 3/4$, and write $a = 3/4(1 + t^2)$ for $t > 0$. If $h_{a,0}$ is positive, then so is

$$H_a(x) := h_{a,0}\left(\frac{2}{3}x, \frac{2}{3}x, \frac{2}{3}x\right) = \frac{1}{1 - 2x + (1 + t^2)x^2}.$$

Observe that

$$t\,x\,H_a(x) = \sum_{n \geqslant 0} \operatorname{im}((1 + i\,t)^n)\,x^n = \sum_{n \geqslant 0} (1 + t^2)^{n/2} \sin(n \arctan(t))\,x^n.$$

We thus see that the Taylor coefficients of H_a change sign infinitely often. In order for $h_{a,0}$ to be positive we therefore need $a \leqslant 3/4$. ☐

Corollary 2. *Let $a \leqslant 3/4$. Then $h_{a,b}$ is positive only if $b \leqslant 2 - 3a + 2(1 - a)^{3/2}$.*

Bibliography

[Str08] Armin Straub. Positivity of Szegö's rational function. *Advances in Applied Mathematics*, 41(2):255–264, 2008.