Partitions with Durfee triangles of fixed size

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Abstract

A well-studied statistic of an integer partition is the size of its Durfee square. In particular, the number $D_k(n)$ of partitions of n with Durfee square of fixed size k has a well-known simple rational generating function. We study the number $R_k(n)$ of partitions of n with Durfee triangle of size k (the largest subpartition with parts $1, 2, \ldots, k$). We determine the corresponding generating functions which are rational functions of a similar form. Moreover, we explicitly determine the leading asymptotic of $R_k(n)$, as $n \to \infty$.

1 Introduction

A partition of an integer $n \geq 0$ is an ordered list $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 1$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_d$. The λ_j are called the parts of λ. For nice introductions to partitions, we refer to [And76] and [AE04]. Recall that the Ferrers board of the partition λ consists of d left-aligned rows such that the j^{th} row contains λ_j boxes. Its Durfee square is the largest square within the Ferrers board. We say that the Durfee square of λ has size k if its sides have length k (in which case it consists of $k \times k$ boxes). Equivalently, k is the largest integer such that $\lambda_j \geq k$ for all $j \in \{1, 2, \ldots, k\}$. Let $D_k(n)$ be the number of partitions of n whose Durfee square has size k. It is well-known that

$$\sum_{n=0}^{\infty} D_k(n) q^n = \frac{q^{k^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^k)^2} \tag{1}$$

(see, for instance, [And76, p. 28] or [AE04, Section 8.1]).

As illustrated in Figure 1, one can likewise consider the *Durfee triangle* of a partition λ as the largest right-angled isosceles triangle within the Ferrers board whose apex is the top-left corner. We say that the Durfee triangle of λ has size k if its horizontal and vertical sides have length k each. Equivalently, k is the largest integer such that $\lambda_j > k - j$ for all $j \in \{1, 2, \ldots, k\}$. This notion of Durfee triangle was recently introduced by the first author [Sha25], who showed that the size of the Durfee triangle of λ equals the maximal number of non-intersecting rooks that can be placed on the Ferrer's board of λ. In the present paper, we study the number $R_k(n)$ of partitions of n whose Durfee triangle has size k. Our first main result is that the generating functions $\mathcal{F}_k(q)$ of $R_k(n)$ take a form that is similar to the generating functions (1) for $D_k(n)$, though more complicated and less explicit.

Theorem 1.1. *For any fixed positive integer k, we have*

$$\mathcal{F}_k(q) := \sum_{n=0}^{\infty} R_k(n) q^n = \frac{q^{k(k+1)/2} \varphi_k(q)}{(1-q)(1-q^2)\cdots(1-q^k)} \tag{2}$$

where

$$\varphi_k(q) = 1 + c_1 q + \cdots + c_{k^2-1} q^{k^2-1} + (-1)^{k-1} q^{k^2}$$

is a polynomial of degree k^2 with integer coefficients. Moreover, if k is odd then $\varphi_k(-1) = 0$.

The initial cases $k \in \{1,2,3,4,5\}$ were individually considered in [Sha25] and Theorem 1.1 extends this to all k in a uniform manner. Theorem 1.1 is proved in Section 4 as Theorem 4.1 (which is stated in terms of the normalized variation $F_k(q) = q^{-k(k+1)/2}\mathcal{F}_k(q)$) and Lemma 4.2. The proof proceeds from an explicit representation of the generating function $\mathcal{F}_k(q)$ as a multiple sum which is discussed in Section 2. This q-multisum is a convolution sum of a related simpler q-series $A_d(q)$ which is further analyzed in Section 3.

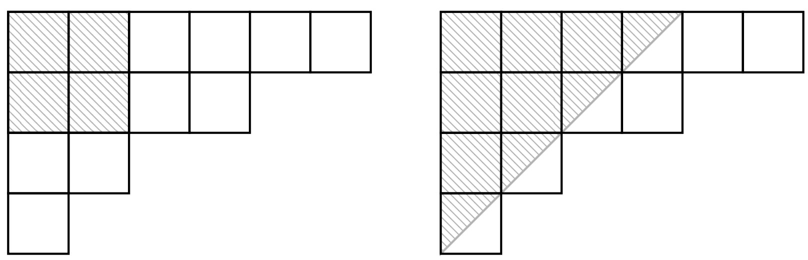

Figure 1: Durfee square and triangle (of size 2 and 4) of $\lambda = (6,4,2,1)$

As a consequence of the generating functions (2), the sequences $R_k(n)$ are constant recursive, or C-finite, of order $k(k+1)/2$ (and the order can be reduced by 1 if $k \geq 3$ is odd). This is discussed in Section 5. It further follows (see Corollary 5.7) that the values $R_k(n)$, for fixed k and all $n > k^2$, are given by a quasi-polynomial of degree $k-1$ and quasi-period $\operatorname{lcm}(1,2,\dots,k)$. Analogously, Choliy and Sills [CS16] deduce from (1) that $D_k(n)$ is a quasi-polynomial of degree $2k-1$ and quasi-period $\operatorname{lcm}(1,2,\dots,k)$. Moreover, they show that the leading terms are

$$D_k(n) = \frac{1}{(k!)^2}\frac{n^{2k-1}}{(2k-1)!} - \frac{1}{2k!(k-2)!}\frac{n^{2k-2}}{(2k-2)!} + O(n^{2k-3}). \tag{3}$$

Additional terms of the asymptotic expansion as $n \to \infty$ can be worked out algorithmically as observed by Sills and Zeilberger [SZ12]. Establishing the asymptotics for $R_k(n)$ is more intricate because the generating functions (2) are less explicit than (1) when working with general k. Our second main result is to determine the leading term in Section 5.

Theorem 1.2. *Fix a positive integer k. As $n \to \infty$, we have*

$$R_k(n) = \frac{2^k}{k!} \frac{n^{k-1}}{(k-1)!} + O(n^{k-2}). \tag{4}$$

It would be desirable to further analyze the generating functions (2) and, as an application, to determine additional terms in the asymptotic expansion of $R_k(n)$. Various other avenues for future work are outlined in the final section.

2 A convolution sum for $\mathcal{F}_k(q)$

We begin by describing the generating function $\mathcal{F}_k(q)$ in terms of a convolution sum involving the following simpler q-series which we study in more detail in the next section:

$$A_d(q) := \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1+1} \cdots \sum_{k_d=0}^{k_{d-1}+1} q^{k_1+k_2+\cdots+k_d} = \sum_{n=0}^{\infty} a_d(n) q^n, \tag{5}$$

where d is a positive integer. We further set $A_0(q) = 1$. Throughout, let

$$T_k = 1 + 2 + \cdots + k = \frac{1}{2} k(k+1)$$

be the k^{th} triangular number.

Lemma 2.1. *We have*

$$\mathcal{F}_k(q) = q^{T_k} \sum_{d=0}^{k} q^d A_d(q) A_{k-d}(q). \tag{6}$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \ldots$ be a partition with Durfee triangle of size k, and let $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ be its conjugate. This implies that $\lambda_j \geq k - j + 1$ (as well as $\lambda'_j \geq k - j + 1$) for $j \in \{1, 2, \ldots, k\}$. Choose $d \in \{0, 1, \ldots, k\}$ to be maximal such that $\lambda_j > k - j + 1$ for all $j \in \{1, 2, \ldots, d\}$, and set $m_j = \lambda_j - (k - j + 1)$. In other words, d is the number of initial rows in the Ferrers diagram of λ that are strictly longer than the Durfee triangle, and $m_j \geq 1$ is the amount by which the length of the j^{th} row exceeds the Durfee triangle. Note that we have $m_1 \geq 1$, $1 \leq m_2 \leq m_1 + 1$, ..., $1 \leq m_d \leq m_{d-1} + 1$.

On the other hand, set $n_j = \lambda'_j - k + j - 1 \geq 0$ for $j \in \{1, 2, \ldots, k\}$ to be the amount by which the length of the j^{th} column in the Ferrers diagram of λ exceeds the Durfee triangle. Note that we have $n_1 \geq 0$, $0 \leq n_2 \leq n_1 + 1$, ..., $0 \leq n_{k-d} \leq n_{k-d-1} + 1$ (and that, by the definition of d, we have $n_{k-d+1} = 0$ provided that $d \neq 0$). See Figure 2 for a diagram illustrating the situation.

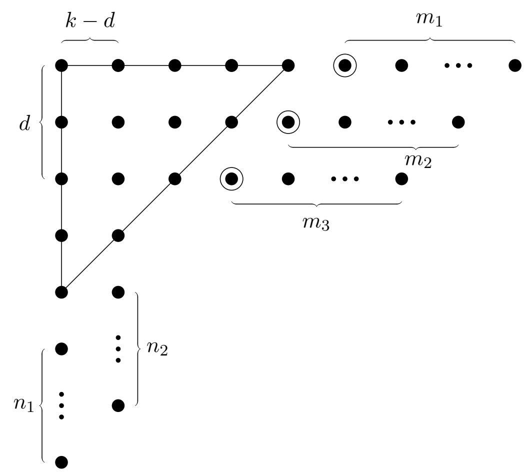

Figure 2: Decomposing a partition according to its Durfee triangle

The Ferrers diagram of λ decomposes into the Durfee triangle, the $m_1+\cdots+m_d$ excess nodes in the first d rows, as well as the $n_1+\cdots+n_{k-d}$ excess nodes in the first $k-d$ columns. It follows that the generating function $\mathcal{F}_k(q)$ for partitions with Durfee triangle of size k is given by

$$q^{T_k}\sum_{d=0}^{k}\left[\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{m_1+1}\cdots\sum_{m_d=1}^{m_{d-1}+1}q^{m_1+\cdots+m_d}\right]\left[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1+1}\cdots\sum_{n_{k-d}=0}^{n_{k-d-1}+1}q^{n_1+\cdots+n_{k-d}}\right].$$

The second bracketed multi-sum is $A_{k-d}(q)$, by the definition of the latter. On the other hand, the first bracketed multi-sum can be rewritten as

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{m_1+1}\cdots\sum_{m_d=1}^{m_{d-1}+1}q^{m_1+\cdots+m_d}=q^d\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1+1}\cdots\sum_{n_d=0}^{n_{d-1}+1}q^{n_1+\cdots+n_d}=q^dA_d(q).$$

Algebraically, this follows by setting $n_i=m_i-1$; combinatorially, this reflects the fact that each of the d rows contains at least one excess node and these d first nodes are peeled off into the overall factor q^d.

In terms of the function $A_d(q)$, we therefore conclude that

$$\mathcal{F}_k(q)=q^{T_k}\sum_{d=0}^{k}q^dA_d(q)A_{k-d}(q),$$

as claimed. □

3 The generating function $A_d(q)$

Let $A_d(q) = \sum_{n=0}^{\infty} a_d(n) q^n$ be as defined in (5). The values of the numbers $a_d(n)$ for small d and n are recorded in the following table:

$d \backslash n$	0	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	3	3	4	4	5	5	6	6
3	1	3	4	6	7	9	11	13	15	18	20
4	1	4	7	11	15	19	25	30	37	44	53
5	1	5	11	19	29	39	53	67	84	103	126
6	1	6	16	31	52	76	107	143	184	233	289
7	1	7	22	48	88	140	207	291	389	508	646
8	1	8	29	71	142	245	384	567	792	1069	1401

By the defining multisum (5), $a_d(n)$ is the number of weak compositions of n into d parts such that each part does not exceed its predecessor by more than 1. In other words, $a_d(n)$ is the number of tuples $(k_1, k_2, \ldots, k_d)$ of nonnegative integers k_j with $k_1 + k_2 + \cdots + k_d = n$ such that $k_j \leq k_{j-1} + 1$ for $j \in \{2, 3, \ldots, d\}$. On the other hand, it follows as in the proof of Lemma 2.1 that $a_d(n)$ equals the number of partitions of $n + T_d$ into exactly d parts with Durfee triangle of size d. In other words, $q^{T_d} A_d(q)$ can be interpreted as the generating function for partitions into d parts with Durfee triangle of size d.

Example 3.1. For instance, $a_3(5) = 9$. The corresponding weak compositions of 5 into 3 parts such that each part does not exceed its predecessor by more than 1 are

$$(5,0,0), (4,1,0), (4,0,1), (3,2,0), (3,1,1), (2,3,0), (2,2,1), (2,1,2), (1,2,2).$$

On the other hand, by adding the Durfee triangle $(3,2,1)$ componentwise to these compositions, we obtain the corresponding partitions of $5 + T_3 = 11$ into 3 parts with Durfee triangle of size 3:

$$(8,2,1), (7,3,1), (7,2,2), (6,4,1), (6,3,2), (5,5,1), (5,4,2), (5,3,3), (4,4,3).$$

Remark 3.2. The sequences $a_d(n)$ for fixed $d \leq 10$ are recorded in the OEIS [OEI25] as the "number of partitions into d parts such that every i^{th} smallest part (counted with multiplicity) is different from i". Here one has to consider partitions of size $n + T_d + d$ and the corresponding partitions can be obtained from those of $n + T_d$ into d parts with Durfee triangle of size d by simply increasing each part by 1.

For small values of d, we can use the multisum (5) to compute $A_d(q)$ explicitly. For instance, in addition to the trivial $A_1(q) = 1/(1-q)$, we find

$$\begin{aligned} A_2(q) &= \frac{1+q-q^2}{(1-q)^2(1+q)}, \\ A_3(q) &= \frac{1+2q-q^3-2q^4+q^6}{(1-q)^3(1+q)(1+q+q^2)}, \\ A_4(q) &= \frac{(1+q-q^4)(1+2q-2q^4-q^5+q^8)}{(1-q)^4(1+q)^2(1+q+q^2)(1+q^2)}. \end{aligned}$$

Firstly, we observe that the denominators of these three rational functions are $(q;q)_2$, $(q;q)_3$ and $(q;q)_4$, respectively, where $(q;q)_d$ is the q-Pochhammer

$$(q;q)_d = \prod_{r=1}^{d} (1-q^r). \tag{7}$$

The remainder of this section focuses on proving the following result which shows, in particular, that this observation extends to $A_d(q)$ in general.

Theorem 3.3. *For all positive integers d,*

$$A_d(q) = \frac{\alpha_d(q)}{(q;q)_d} \tag{8}$$

where $\alpha_d(q) \in 1 + q\mathbb{Z}[q]$ is a polynomial of degree $(d-1)d$. Moreover, the coefficient of $q^{(d-1)d}$ in $\alpha_d(q)$ is $(-1)^{d-1}$.

Numerical evidence suggests that Theorem 3.3 is best possible in the sense that the quotient on the right-hand side of (8) is in lowest terms and cannot be further reduced. We have confirmed this for all $d \le 20$ by explicitly calculating $A_d(q)$.

In order to prove Theorem 3.3, we first give a recursive characterization of $A_d(q)$. To this end, we introduce the auxiliary generating functions

$$A_d(x_1, \dots, x_d) := \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1+1} \cdots \sum_{k_d=0}^{k_{d-1}+1} x_1^{k_1} \cdots x_d^{k_d}. \tag{9}$$

We note that there is no harm in denoting the sums in (5) and (9) both with A_d because they are equal in the case $d = 1$. More generally, we have $A_d(q) = A_d(q, \dots, q)$, where the right-hand side features d copies of q.

Combined with the base case $A_1(x_1) = 1/(1-x_1)$, the following recursively determines the series $A_d(x_1, \dots, x_d)$ for all positive integers d.

Lemma 3.4. *For integers $d > 1$,*

$$A_d(x_1, \dots, x_d) = \frac{A_{d-1}(x_1, \dots, x_{d-1}) - x_d^2 A_{d-1}(x_1, \dots, x_{d-2}, x_{d-1}x_d)}{1-x_d}. \tag{10}$$

Proof. This follows directly from the definition (9) of $A_d(x_1,\ldots,x_d)$ by evaluating the innermost geometric series:

$$\begin{aligned} A_d(x_1,\ldots,x_d) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1+1}\cdots\sum_{k_{d-1}=0}^{k_{d-2}+1} x_1^{k_1}\cdots x_{d-1}^{k_{d-1}} \sum_{k_d=0}^{k_{d-1}+1} x_d^{k_d} \\ &= \frac{1}{1-x_d}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{k_1+1}\cdots\sum_{k_{d-1}=0}^{k_{d-2}+1} x_1^{k_1}\cdots x_{d-1}^{k_{d-1}}(1-x_d^{k_{d-1}+2}) \\ &= \frac{A_{d-1}(x_1,\ldots,x_{d-1}) - x_d^2 A_{d-1}(x_1,\ldots,x_{d-2},x_{d-1}x_d)}{1-x_d}. \end{aligned}$$

□

It is clear from this recursive description that $A_d(x_1,\ldots,x_d)$ is a rational function in the variables $x_1,\ldots,x_d$. Moreover, we can describe the denominators as follows.

Lemma 3.5. *For all positive integers d,*

$$A_d(x_1,\ldots,x_d) = \frac{\alpha_d(x_1,\ldots,x_d)}{\prod_{r=1}^{d}(1-x_1x_2\cdots x_r)}, \tag{11}$$

where $\alpha_d(x_1,\ldots,x_d)\in\mathbb{Z}[x_1,\ldots,x_d]$ is a polynomial of degree $d-1$ in each variable. Moreover, $\alpha_d(0,\ldots,0)=1$ and the coefficient of $(x_1\cdots x_d)^{d-1}$ is $(-1)^{d-1}$.

Proof. Since $A_1(x_1)=1/(1-x_1)$, the claim is true for $d=1$. For the purpose of induction on d, we fix d and assume that (11) has already been shown to hold for $d-1$. In particular, we may assume that

$$\begin{aligned} A_{d-1}(x_1,\ldots,x_{d-1}) &= \frac{\alpha_{d-1}(x_1,\ldots,x_{d-1})}{\prod_{r=1}^{d-1}(1-x_1x_2\cdots x_r)}, \\ A_{d-1}(x_1,\ldots,x_{d-2},x_{d-1}x_d) &= \frac{\alpha_{d-1}(x_1,\ldots,x_{d-2},x_{d-1}x_d)}{(1-x_1x_2\cdots x_{d-1}x_d)\prod_{r=1}^{d-2}(1-x_1x_2\cdots x_r)}. \end{aligned}$$

Note that the least common multiple of the two denominators on the right-hand sides is $\prod_{r=1}^{d}(1-x_1x_2\cdots x_r)$. By Lemma 3.4, we therefore have

$$\begin{aligned} A_d(x_1,\ldots,x_d) &= \frac{A_{d-1}(x_1,\ldots,x_{d-1}) - x_d^2 A_{d-1}(x_1,\ldots,x_{d-2},x_{d-1}x_d)}{1-x_d} \\ &= \frac{\beta_d(x_1,\ldots,x_d)}{(1-x_d)\prod_{r=1}^{d}(1-x_1x_2\cdots x_r)}, \end{aligned} \tag{12}$$

for some $\beta_d(x_1,\ldots,x_d)\in\mathbb{Z}[x_1,\ldots,x_d]$. On the other hand, observe that

$$A_{d-1}(x_1,\ldots,x_{d-1}) - x_d^2 A_{d-1}(x_1,\ldots,x_{d-2},x_{d-1}x_d)$$

vanishes if we set $x_d = 1$. It follows that $\beta_d(x_1,\ldots,x_d)$ is a multiple of $1-x_d$, and this cancels with the factor $1-x_d$ in the denominator of (12), resulting in (11) with $\alpha_d(x_1,\ldots,x_d) = \beta_d(x_1,\ldots,x_d)/(1-x_d) \in \mathbb{Z}[x_1,\ldots,x_d]$.

Since the constant term of the denominator of (11) is 1, it follows from the definition (9) that the constant term $\alpha_d(0,\ldots,0)$ of the numerator is 1 as well. Finally, to extract the numerator term of highest degree, we note that

$$\begin{aligned}\beta_d(x_1,\ldots,x_d) \quad = \quad & (1-x_1x_2\cdots x_d)\alpha_{d-1}(x_1,\ldots,x_{d-1}) \\ & -x_d^2(1-x_1x_2\cdots x_{d-1})\alpha_{d-1}(x_1,\ldots,x_{d-2},x_{d-1}x_d).\end{aligned}$$

It then follows from the induction hypothesis that the term of highest degree in $\beta_d(x_1,\ldots,x_d)$ is

$$-x_d^2(-x_1x_2\cdots x_{d-1})(-1)^{d-2}(x_1x_2\cdots x_d)^{d-2} = x_d(-1)^{d-2}(x_1x_2\cdots x_d)^{d-1}.$$

Consequently, after dividing $\beta_d(x_1,\ldots,x_d)$ by the factor $1-x_d$, we find that the term of highest degree in $\alpha_d(x_1,\ldots,x_d)$ is $(-1)^{d-1}(x_1x_2\cdots x_d)^{d-1}$ as claimed. □

Since $A_d(q) = A_d(q,\ldots,q)$, we observe that Lemma 3.5 implies Theorem 3.3 as a special case as shown below.

Proof of Theorem 3.3. In the special case where we set all variables $x_1,\ldots,x_d$ to q, the product $\prod_{r=1}^{d}(1-x_1x_2\cdots x_r)$ becomes the q-Pochhammer $(q;q)_d$. As such, Lemma 3.5 shows that

$$A_d(q) = \frac{\alpha_d(q)}{(q;q)_d}, \quad \alpha_d(q) = \alpha_d(q,\ldots,q) \in 1 + q\mathbb{Z}[q],$$

as claimed in (8). Moreover, the fact that the term of highest degree in $\alpha_d(x_1,\ldots,x_d)$ is $(x_1\cdots x_d)^{d-1}$ implies that the degree of $\alpha_d(q)$ is $(d-1)d$. □

We conclude this section by establishing the following asymptotic result for $a_d(n)$ which we will use in Section 5 for determining the asymptotics of $R_k(n)$ as $n\to\infty$.

Lemma 3.6. *Fix a positive integer d. As $n\to\infty$, we have*

$$a_d(n) = \frac{1}{d!}\frac{n^{d-1}}{(d-1)!} + O(n^{d-2}). \tag{13}$$

Proof. Let $p_d(n)$ denote the number of partitions of n into at most d parts. Since $a_d(n)$ counts weak compositions of n into d parts such that each part does not exceed its predecessor by more than 1, we clearly have $p_d(n) \le a_d(n)$. On the other hand, we know from the proof of Lemma 2.1 that $a_d(n)$ equals the number of partitions of $n+T_d$ into exactly d parts with Durfee triangle of size d. This implies that $a_d(n) \le p_d(n+T_d)$. Combined, we therefore have

$$p_d(n) \le a_d(n) \le p_d(n+T_d). \tag{14}$$

On the other hand, it is well-known that

$$p_d(n) = \frac{1}{d!}\frac{n^{d-1}}{(d-1)!} + O(n^{d-2}). \tag{15}$$

We refer, for instance, to [PS78, Part 1, Problem 27] or [RA05, Theorem 4.2.1] where the result is attributed to Laguerre and Schur. The bounds (14) combined with (15) show that $a_d(n) \sim p_d(n)$ as $n \to \infty$. This proves (13).

For completeness, we indicate a direct proof of (15). To that end, observe that the argument used in the proof of Theorem 5.4 also applies to the coefficients $c_d(n)$ of

$$\frac{\gamma_d(q)}{(q;q)_d} = \sum_{n=0}^{\infty} c_d(n) q^n,$$

where $\gamma_d(q)$ is some polynomial in q, and shows that, as $n \to \infty$,

$$c_d(n) = \frac{\gamma_d(1)}{d!}\frac{n^{d-1}}{(d-1)!} + O(n^{d-2}). \tag{16}$$

Since the numbers $p_d(n)$ have the simple generating function

$$\sum_{n\geq 0} p_d(n) q^n = \frac{1}{(q;q)_d}, \tag{17}$$

the claimed asymptotics (15) follows from (16) upon setting $\gamma_d(q) = 1$. □

Corollary 3.7. *Let $\alpha_d(q) = A_d(q)(q;q)_d$ as in (8). Then, for all positive integers d, we have $\alpha_d(1) = 1$.*

Proof. Setting $\gamma_d(q) = \alpha_d(q)$ in (16), it follows that

$$a_d(n) = \frac{\alpha_d(1)}{d!}\frac{n^{d-1}}{(d-1)!} + O(n^{d-2}).$$

Comparison with (13) then shows that $\alpha_d(1) = 1$. □

4 The generating function $\mathcal{F}_k(q)$

We now return to the generating function $\mathcal{F}_k(q) \in q^{T_k}\mathbb{Z}[[q]]$. To simplify exposition, we consider the normalization

$$F_k(q) = q^{-T_k}\mathcal{F}_k(q) = \sum_{n=0}^{\infty} f_k(n) q^n \in 1 + q\mathbb{Z}[[q]],$$

so that $R_k(n) = f_k(n - T_k)$. The values of the numbers $f_k(n)$ for small k and n are recorded in the following table (cf. A325188 in [OEI25]):

$k\backslash n$	0	1	2	3	4	5	6	7	8	9	10
1	1	2	2	2	2	2	2	2	2	2	2
2	1	3	5	8	9	12	13	16	17	20	21
3	1	4	8	15	23	32	43	54	67	82	97
4	1	5	12	24	42	66	98	135	181	233	298
5	1	6	17	37	70	118	189	282	402	552	736
6	1	7	23	55	112	201	337	533	801	1158	1617
7	1	8	30	79	173	331	581	959	1502	2262	3286
8	1	9	38	110	259	528	974	1676	2724	4241	6368

In the previous section, we observed that $A_d(q)$ is a rational function in q. In light of Lemma 2.1, it follows that $F_k(q)$, and hence $\mathcal{F}_k(q)$, are rational functions as well. For instance, from the expressions for $A_d(q)$ given in the previous section for $d \in \{1, 2, 3, 4\}$ we readily find that $F_1(q) = (1 + q)/(1 - q)$ as well as

$$\begin{aligned} F_2(q) &= \frac{1 + 2q + q^2 + q^3 - q^4}{(1 - q)^2(1 + q)}, \\ F_3(q) &= \frac{1 + 2q + q^2 + 2q^3 - q^4 - q^6 - q^7 + q^8}{(1 - q)^3(1 + q + q^2)}, \\ F_4(q) &= \frac{1 + 4q + 6q^2 + 7q^3 + 6q^4 + 2q^5 - 5q^7 - 5q^8 - 5q^9 + q^{11} + 3q^{12} + 2q^{13} - q^{16}}{(1 - q)^4(1 + q)^2(1 + q + q^2)(1 + q^2)} \end{aligned}$$

which match the formulas derived in [Sha25] (also see the entries A325168, A382682, A384562 in [OEI25]). The denominators of the latter three rational functions are $(q; q)_2$, $(q; q)_3/(1 + q)$ and $(q; q)_4$, respectively. If written as rational functions with denominator $(q; q)_k$, the numerator degrees of $F_k(q)$ are k^2 for $k \in \{1, 2, 3, 4\}$. The next result proves that this observation holds in general. In Lemma 4.2, we then show that the denominator can be reduced to $(q; q)_k/(1 + q)$ for all odd $k \geq 3$.

Theorem 4.1. *For all positive integers k,*

$$F_k(q) = \frac{\varphi_k(q)}{(q; q)_k} \tag{18}$$

where $\varphi_k(q) \in 1 + q\mathbb{Z}[q]$ is a polynomial of degree k^2. Moreover, the coefficient of q^{k^2} in $\varphi_k(q)$ is $(-1)^{k-1}$.

Proof. By Lemma 2.1 combined with Theorem 3.3, we have

$$F_k(q) = \sum_{d=0}^{k} q^d A_d(q) A_{k-d}(q) = \sum_{d=0}^{k} q^d \frac{\alpha_d(q)}{(q; q)_d} \frac{\alpha_{k-d}(q)}{(q; q)_{k-d}}, \tag{19}$$

for polynomials $\alpha_d(q) \in 1 + q\mathbb{Z}[q]$ of degree $(d-1)d$. This true also in the case $d = 0$, for which we have $A_0(q) = 1$ and, hence, $\alpha_0(q) = 1$. Recall the well-known fact that the quotients

$$\binom{k}{d}_q := \frac{(q;q)_k}{(q;q)_d (q;q)_{k-d}} \tag{20}$$

are polynomials in $\mathbb{Z}[q]$, known as the Gaussian polynomials or q-binomial coefficients. We conclude from (19) that $F_k(q) = \varphi_k(q)/(q;q)_k$ where

$$\varphi_k(q) = \sum_{d=0}^{k} \binom{k}{d}_q q^d \alpha_d(q) \alpha_{k-d}(q) \in 1 + q\mathbb{Z}[q]. \tag{21}$$

It follows readily from (7) that $(q;q)_d$ has degree $d(d+1)/2$. Accordingly, the q-binomial coefficients in (20) have degree $d(k-d)$. Using further that $\alpha_d(q)$ has degree $(d-1)d$, standard calculations show that, in the sum for $\varphi_k(q)$, the summand corresponding to $d = k$ is the unique one contributing the term of highest degree in q. That summand simplifies to $q^k \alpha_k(q)$ since the involved q-binomial coefficient as well as $\alpha_0(q)$ equal 1. Therefore, the degree of $\varphi_k(q)$ equals the degree of $q^k \alpha_k(q)$ which is $k + (k-1)k = k^2$. Moreover, by Theorem 3.3, the coefficient of q^{k^2} is $(-1)^{k-1}$. □

While the denominators for $A_d(q)$ provided by Theorem 3.3 appear best possible, numerical evidence suggests that, as we observed in the case $F_3(q)$ above, the denominators for $F_k(q)$ provided by Theorem 4.1 can be improved by removing the factor $1+q$ whenever $k \geq 3$ is odd.

Lemma 4.2. *Write $F_k(q) = \varphi_k(q)/(q;q)_k$ as in (18). If $k \geq 1$ is odd then*

$$\varphi_k(q) \in (1+q)\mathbb{Z}[q]. \tag{22}$$

Proof. From (21), we have

$$\varphi_k(q) = \sum_{d=0}^{k} \binom{k}{d}_q q^d \alpha_d(q) \alpha_{k-d}(q) = q^k \sum_{d=0}^{k} \binom{k}{d}_q q^{-d} \alpha_d(q) \alpha_{k-d}(q) \tag{23}$$

where, for the second equality, we replaced d by $k-d$ in the summand and used the symmetry

$$\binom{k}{k-d}_q = \binom{k}{d}_q$$

of the q-binomial coefficients. Upon setting $q = -1$ in both of the sums in (23) and comparing the results, we find that

$$\varphi_k(-1) = (-1)^k \varphi_k(-1).$$

If k is odd then this implies that $\varphi_k(-1) = 0$ which is equivalent to (22). □

Apart from this observation, numerical evidence suggests that Theorem 4.1 is best possible. In other words, it appears that the minimal denominator of $F_k(q)$ equals $(q;q)_k$, if k is even, and equals $(q;q)_k/(1+q)$ if $k \geq 3$ is odd (since, for $k \geq 2$, the factor $1+q$ divides both $(q;q)_k$ and $\varphi_k(q)$). By explicitly calculating $F_k(q)$, we have confirmed this conjecture for $k \leq 20$.

Example 4.3. The first case not covered by the approach in [Sha25] is $k = 6$. In that case, in accordance with Theorem 4.1, we find $F_6(q) = \varphi_6(q)/(q;q)_6$ where $\varphi_6(q)$ is the degree 6^2 polynomial

$$\begin{aligned}\varphi_6(q) \quad = \quad & 1 + 6q + 15q^2 + 25q^3 + 34q^4 + 35q^5 + 31q^6 + 20q^7 - 19q^9 - 39q^{10} \\ & -48q^{11} - 50q^{12} - 36q^{13} - 19q^{14} + 13q^{15} + 30q^{16} + 45q^{17} + 42q^{18} \\ & +28q^{19} + 11q^{20} - 8q^{21} - 21q^{22} - 24q^{23} - 15q^{24} - 9q^{25} + 2q^{26} + 3q^{27} \\ & +5q^{28} + 3q^{29} + 2q^{30} + 2q^{31} - q^{36}.\end{aligned}$$

5 Asymptotics and exact formulas for $R_k(n)$

Recall that a sequence $a(n)$ is *constant recursive*, or C-finite, of order r if there exist complex numbers $c_0, c_1, \ldots, c_{r-1}$ such that

$$a(n+r) = c_{r-1}a(n+r-1) + \cdots + c_1 a(n+1) + c_0 a(n) \tag{24}$$

for all $n \geq n_0$. The corresponding polynomial

$$P(x) = x^r - c_{r-1}x^{r-1} - \cdots - c_1 x - c_0 \tag{25}$$

is called the characteristic polynomial. We refer to [EvdPSW03] or [KP11, Chapter 4] for introductions to constant recursive sequences. It is well-known that constant recursive sequences are exactly those whose generating function is rational. More precisely, the condition that $a(n)$ satisfies a recurrence of the form (24) for all $n \geq n_0$ is equivalent to

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{N(q)}{D(q)} \tag{26}$$

for polynomials $N, D \in \mathbb{C}[q]$, and $\deg(N) < \deg(D) + n_0$. Moreover, if $P(x)$ is the characteristic polynomial (25) of the recurrence then we can choose

$$D(q) = q^r P(1/q) = 1 - c_{r-1}q - \cdots - c_1 q^{r-1} - c_0 q^r.$$

It therefore follows from Theorem 4.1 that, for every fixed positive integer k, the sequence $R_k(n)$ is constant recursive. More precisely, we find the following:

Lemma 5.1. *Fix a positive integer k. The sequence $R_k(n)$ is constant recursive and, for all $n > k^2$, satisfies a recurrence (24) of order*

$$r = \begin{cases} T_k - 1, & \text{if } k \geq 3 \text{ is odd,} \\ T_k, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 4.1,

$$\mathcal{F}_k(q) = \sum_{n=0}^{\infty} R_k(n) q^n = q^{T_k} \frac{\varphi_k(q)}{(q;q)_k} \tag{27}$$

where $\varphi_k(q) \in 1 + q\mathbb{Z}[q]$ has degree k^2. As in (26), this implies that $R_k(n)$ is constant recursive. Moreover, since $(q;q)_k$ has degree $1+2+\cdots+k = T_k$, we find that we can choose $n_0 = k^2 + 1$ in the above discussion. The recurrence corresponding to the denominators $(q;q)_k$ has order T_k. On the other hand, we have shown in Lemma 4.2 that, for odd $k \geq 3$, the denominators can be reduced to $(q;q)_k/(1+q)$. Accordingly, for odd $k \geq 3$, the order of the recurrence can be reduced to $T_k - 1$. □

As in the discussion after Lemma 4.2, it appears that the orders of the recurrences in Lemma 5.1 are minimal.

Example 5.2. We have

$$\mathcal{F}_3(q) = \sum_{n=1}^{\infty} R_3(n) q^n = q^{T_3} F_3(q) = \frac{q^6(1 + 2q + q^2 + 2q^3 - q^4 - q^6 - q^7 + q^8)}{(1-q)^3(1+q+q^2)}.$$

Let $P(q) = (1-q)^3(1+q+q^2) = 1 - 2q + q^2 - q^3 + 2q^4 - q^5$. It therefore follows as observed for (26) that the sequence $R_3(n)$ satisfies the recurrence

$$R_3(n+5) = 2R_3(n+4) - R_3(n+3) + R_3(n+2) - 2R_3(n+1) + R_3(n)$$

of order $5 = T_3 - 1$ for all $n > 14 - 5 = 3^2$. This coincides with the conclusion in Corollary 3.3 of [Sha25].

Example 5.3. It likewise follows from Theorem 3.3 that the coefficients $a_d(n)$ of $A_d(q)$ discussed in Section 3 are constant recursive and, for all $n > (d-1)d - T_d$ or, equivalently, $n \geq T_{d-2}$, satisfy a recurrence (24) of order T_d. This proves, in particular, the generating functions and recurrences that are presently listed as conjectured by Chai Wah Wu for the cases $k \in \{4, 5, 6, 7\}$ in the OEIS [OEI25] (sequences `A244240`, `A244241`, `A244242`, `A244243`).

Let $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathbb{C}$ be the roots, with multiplicities $m_1, m_2, \ldots, m_s$, of the characteristic polynomial (25). Then the sequence $a(n)$ admits an exact formula of the form

$$a(n) = \sum_{j=1}^{s} \sum_{r=0}^{m_j - 1} c_{j,r} n^r \lambda_j^n, \tag{28}$$

which is valid for all $n \geq n_0$ (where, as before, n_0 is such that the recurrence (24) holds for all $n \geq n_0$), and where $c_{j,r} \in \mathbb{C}$ are constants. This can be seen, for instance, from the generalized binomial expansion applied to the partial fraction decomposition of (26).

In our case of $\mathcal{F}_k(q) = \sum_{n \geq 1} R_k(n) q^n$, it follows from Theorem 4.1 that the characteristic roots are all roots of unity. Therefore, as $n \to \infty$, the asymptotically dominating contribution comes from the root with the highest multiplicity. Pursuing this line of reasoning ultimately results in the asymptotics claimed in Theorem 1.2 which is restated here. This in particular proves, as conjectured in [Sha25], that $R_k(n) \sim C_k n^{k-1}$ as $n \to \infty$ for suitable constants C_k.

Theorem 5.4. *Fix a positive integer k. As $n \to \infty$, we have*

$$R_k(n) = \frac{2^k}{k!} \frac{n^{k-1}}{(k-1)!} + O(n^{k-2}). \tag{29}$$

Proof. By Theorem 4.1, we have

$$\mathcal{F}_k(q) = \sum_{n=1}^{\infty} R_k(n) q^n = q^{T_k} \frac{\varphi_k(q)}{(q;q)_k}$$

with $\varphi_k(q) \in 1 + q\mathbb{Z}[q]$. We can see from (7) that the roots of $(q;q)_k$ are the roots of unity ζ of order m with $m \leq k$ and their multiplicity is $\lfloor k/m \rfloor$. In other words,

$$(q;q)_k = (-1)^k \prod_{m=1}^{k} \Phi_m^{\lfloor k/m \rfloor}(q)$$

where Φ_m is the m^{th} cyclotomic polynomial (the monic polynomial whose roots are precisely the m^{th} primitive roots of unity). We write $\lambda_1 = 1$ and let $\lambda_2, \lambda_3, \ldots, \lambda_s$ be the other roots of unity of order up to k. We denote with $m_1, m_2, \ldots$ the corresponding multiplicities. Note that $m_1 = k$ and that $m_j \leq k/2$ for $j > 1$. The exact formula (28) therefore implies

$$R_k(n) = \sum_{r=0}^{k-1} c_{1,r} n^r + \sum_{j=2}^{s} \sum_{r=0}^{m_j - 1} c_{j,r} n^r \lambda_j^n = \sum_{r=0}^{k-1} c_{1,r} n^r + O(n^{\lfloor k/2 \rfloor - 1}). \tag{30}$$

It remains to determine the leading coefficient $c_{1,k-1}$. To this end, we write

$$\mathcal{F}_k(q) = q^{T_k} \frac{\varphi_k(q)}{(q;q)_k} = \frac{\tilde{N}(q)}{(1-q)^k \tilde{D}(q)}$$

for polynomials $\tilde{N}(q), \tilde{D}(q) \in \mathbb{Z}[q]$ with $\tilde{N}(1) \neq 0$ and $\tilde{D}(1) \neq 0$. Note that $\tilde{N}(q) = q^{T_k} \varphi_k(q)$ and $\tilde{D}(q) = (q;q)_k/(1-q)^k$. Performing part of the partial fraction expansion, we then have

$$\mathcal{F}_k(q) = \frac{\gamma_k}{(1-q)^k} + \frac{\gamma_{k-1}}{(1-q)^{k-1}} + \cdots + \frac{\gamma_1}{1-q} + \frac{M(q)}{\tilde{D}(q)}$$

where γ_j are constants and $M(q) \in \mathbb{Z}[q]$. Using

$$\frac{(j-1)!}{(1-q)^j} = \sum_{n\geq 0} (n+j-1)(n+j-2)\cdots(n+1)q^n,$$

the coefficients $c_{1,r}$ in (30) can be obtained from the coefficients γ_j. In particular, the leading coefficient can be computed as

$$c_{1,k-1} = \frac{\gamma_k}{(k-1)!} = \frac{1}{(k-1)!}\frac{\tilde{N}(1)}{\tilde{D}(1)}.$$

Since $\tilde{N}(q) = q^{T_k}\varphi_k(q)$, we have $\tilde{N}(1) = \varphi_k(1)$. On the other hand, we find that $\tilde{D}(1) = k!$ because

$$\tilde{D}(q) = \frac{(q;q)_k}{(1-q)^k} = \prod_{r=1}^{k} \frac{1-q^r}{1-q} = \prod_{r=1}^{k}(1+q+q^2+\cdots+q^{r-1}).$$

Combined, we obtain

$$c_{1,k-1} = \frac{1}{(k-1)!}\frac{\tilde{N}(1)}{\tilde{D}(1)} = \frac{\varphi_k(1)}{(k-1)!k!}.$$

By (30), this proves the claimed asymptotics (29) if we can show that $\varphi_k(1) = 2^k$. Indeed, we can conclude from $\alpha_d(1) = 1$, established in Corollary 3.7, combined with (21) that

$$\varphi_k(1) = \sum_{d=0}^{k}\binom{k}{d}\alpha_d(1)\alpha_{k-d}(1) = \sum_{d=0}^{k}\binom{k}{d} = 2^k,$$

as desired. □

Remark 5.5. Note that we obtain the same leading asymptotics for the coefficients $f_k(n)$ of $F_k(q) = q^{-T_k}\mathcal{F}_k(q)$. Namely,

$$f_k(n) = \frac{2^k}{k!}\frac{n^{k-1}}{(k-1)!} + O(n^{k-2}).$$

Recall that a sequence $f(n)$ is a quasi-polynomial of degree m and quasi-period δ if it can be expressed as

$$f(n) = \varphi_m(n)n^m + \varphi_{m-1}(n)n^{m-1} + \cdots + \varphi_0(n)$$

where each $\varphi_j(n)$ has period δ (that is, $\varphi_j(n+\delta) = \varphi_j(n)$). See, for instance, Section 4.4 in [Sta97]. Equivalently, $f(n)$ is a quasi-polynomial of degree m and quasi-period δ if there exist polynomials $f_0(n), f_1(n), \ldots, f_{\delta-1}(n)$ such that $f(n) \equiv f_j(n)$ if $n \equiv j \pmod{\delta}$.

Suppose that each root λ_j of the characteristic polynomial (25) is a root of unity of order δ_j (so that $\lambda_j^{\delta_j} = 1$). Let δ be the least common multiple of $\delta_1, \delta_2, \ldots, \delta_s$, and let m

be the maximum of $m_1, m_2, \ldots, m_s$. Then $a(n)$ is a quasi-polynomial of degree $m-1$ and quasi-period δ. Indeed, it follows from (28) that, for each $\nu \in \{0, 1, \ldots, \delta-1\}$, we have

$$a(\delta n+\nu) = \sum_{j=1}^{s} \sum_{r=0}^{m_j-1} c_{j,r} (\delta n+\nu)^r \lambda_j^{\nu}, \tag{31}$$

providing an explicit formula for $a(\delta n+\nu)$, for all $n \geq n_0$, as a polynomial in n.

Example 5.6. As detailed in [Sha25], James Sellers pointed out that it follows from the rational generating function for $R_3(n)$ given in Example 5.2 that

$$R_3(n) \quad = \quad \begin{cases} 6m^2 - 15m + 7, & \text{if } n = 3m, \\ 6m^2 - 11m + 2, & \text{if } n = 3m+1, \\ 6m^2 - 7m - 1, & \text{if } n = 3m+2, \end{cases}$$

provided that $n > 9$. We note that the right-hand side can be equivalently expressed as

$$R_3(n) = \frac{2}{3} n^2 - 5n + \frac{59}{9} + \frac{4}{9} \cos\left(\frac{2n\pi}{3}\right).$$

In particular, the corresponding first-order asymptotics $R_3(n) \sim \frac{2}{3} n^2$ matches the conclusion of Theorem 5.4 where it is shown that $R_k(n) \sim \frac{2^k}{k!} \frac{n^{k-1}}{(k-1)!}$. Likewise, in the special case $k = 4$, we obtain $R_4(n) \sim \frac{1}{9} n^3$ as shown in [Sha25].

Representations as in the previous example can be worked out for $R_k(n)$ for all fixed k, as a consequence of Theorem 4.1.

Corollary 5.7. *Let k be a positive integer. Then there exists a quasi-polynomial $Q_k(n)$ of degree $k-1$ and quasi-period* $\operatorname{lcm}(1, 2, \ldots, k)$ *such that $R_k(n) = Q_k(n)$ for all $n > k^2$. For $k = 3$, the quasi-period is further reduced to* 3.

Proof. Recall that the roots of the q-Pochhammer $(q;q)_k$ are all the roots of unity of order up to k. More precisely, if μ_m denotes the primitive m^{th} roots of unity, then each $\zeta \in \mu_m$ is a root of $(q;q)_k$ of multiplicity $\lfloor k/m \rfloor$. Accordingly, it follows from Lemma 5.1 and the above discussion that $R_k(n)$ has a representation (28) which here takes the form

$$R_k(n) = \sum_{m=1}^{k} \sum_{\zeta \in \mu_m} \sum_{r=0}^{\lfloor k/m \rfloor - 1} c_{j,r} n^r \zeta^n \tag{32}$$

and is valid for all $n > k^2$. In particular, as for (31), the right-hand side of (32) is a quasi-polynomial of degree $k-1$ and quasi-period $\operatorname{lcm}(1, 2, \ldots, k)$.

The reduction in Lemma 4.2 implies that the maximum value of r for $m = 2$ in (32) gets reduced by 1. This lowers the quasi-period for $k = 3$ from $\operatorname{lcm}(2, 3) = 6$ to 3 but does not affect the quasi-period for other values of k. ◻

As observed by Andrews [And03], quasi-polynomials can always be expressed via a single polynomial formula if we permit use of the floor function. For instance, the piecewise formula in Example 5.6 is equivalent to the alternative representation

$$R_3(n) = \frac{1}{3}(2n-3)(n-7) + \frac{2}{3}\left\lfloor \frac{n+1}{3} \right\rfloor + \frac{4}{3}\left\lfloor \frac{n}{3} \right\rfloor$$

for $n > 9$. The process of using computer algebra to automatically obtain quasi-polynomial representations from an appropriate generating function is described by Sills and Zeilberger [SZ12]. The discussion in [SZ12] is focused on the examples of $p_m(n)$, the number of partitions of n into at most m parts, as well as $D_k(n)$, the number of partitions of n whose Durfee square has size k. The approach, however, applies equally to the numbers $R_k(n)$ in light of the generating function provided by Theorem 4.1.

6 Conclusions and future work

In Theorem 4.1 we showed that the generating function $\mathcal{F}_k(q)$ of the number $R_k(n)$ of partitions of n with Durfee triangle of fixed size k takes the form $q^{T_k}\varphi_k(q)/(q;q)_k$. This is derived from the q-multisum representation (6). Neither of these is as explicit as one would like for certain applications such as extracting the asymptotic expansions of $R_k(n)$, as $n \to \infty$, to higher order. For instance, computing the leading order asymptotics in Theorem 5.4 required showing that the numerator polynomials $\varphi_k(q)$ satisfy $\varphi_k(1) = 2^k$. It would be desirable to determine the polynomials $\varphi_k(q)$ more explicitly.

By the relationship (21), information on $\varphi_k(q)$ can be inferred from corresponding information on the simpler polynomials $\alpha_d(q)$ defined by (8). The mentioned evaluations of $\varphi_k(1)$, for instance, follow from the simpler evaluations $\alpha_d(1) = 1$ that we showed in Corollary 3.7. More generally, it appears worthwhile to investigate the values of the polynomials $\alpha_d(q)$ and $\varphi_k(q)$ at roots of unity. For instance, it appears that $\alpha_d(-1) = (-1)^{\lfloor d/2 \rfloor}$ for all integers $d \geq 0$. By (21), this implies the corresponding values

$$\varphi_k(-1) = \begin{cases} (-2)^{k/2}, & \text{if } k \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

The fact that $\varphi_k(-1) = 0$ for odd k follows from Lemma 4.2 according to which $1 + q$ divides $\varphi_k(q)$ for odd k. Also, apart from these cases and the sporadic case

$$\alpha_4(q) = (1 + q - q^4)(1 + 2q - 2q^4 - q^5 + q^8),$$

the polynomials $\alpha_d(q)$ and $\varphi_k(q)$ appear to be irreducible.

In a related direction, it would be of interest to obtain formulas for the coefficients that arise in the (ordinary) partial fraction decomposition of $\mathcal{F}_k(q)$ or its q-partial fraction decomposition in the sense of Munagi [Mun07] or, possibly, a partial fraction-type expansion in the spirit of MacMahon [Sil19]. In the case of $p_d(n)$, Rademacher [Rad73, Section 130] considered the partial fraction decomposition

$$\frac{1}{(q;q)_d} = \sum_{k=1}^{d} \sum_{\substack{0\le h<k \\ (h,k)=1}} \sum_{\ell=1}^{\lfloor d/k \rfloor} \frac{C_{h,k,\ell}(d)}{(q-e^{2\pi i h/k})^{\ell}}$$

and asked for formulas for the coefficients $C_{h,k,\ell}(d)$. Andrews [And03] provided first instances of such formulas but they are involved and not convenient for computations. Sills and Zeilberger [SZ13] showed that certain coefficients $C_{h,k,\ell}(d)$ can be efficiently computed in a recursive manner. Rademacher conjectured that the coefficients $C_{h,k,\ell}(d)$ converge as $d \to \infty$. This was proved false by Drmota and Gerhold [DG14] and, independently, O'Sullivan [O'S16] after Sills and Zeilberger [SZ13] provided strong numerical evidence that Rademacher's conjecture is false. On the other hand, it remains an open problem to clarify the relationship between the partial fraction decompositions of $1/(q;q)_d$ and the limiting case $1/(q;q)_\infty$ which admits a version of a partial fraction decomposition by the Hardy–Ramanujan–Rademacher formula for $p(n)$. For instance, as asked in [DG14], do the coefficients $C_{h,k,\ell}(d)$ converge in a weaker generalized sense as $d \to \infty$? These questions may also be considered if one replaces (ordinary) partial fraction decompositions with the q-partial fraction decomposition in the sense of Munagi [Mun07]. In the special case $h = k = 1$, Munagi [Mun08] proves that the analog of Rademacher's conjecture actually holds true. It would be valuable to extend this analysis to other coefficients as well as to similarly investigate the generating functions studied in this paper.

Recall that every constant recursive integer sequence is necessarily eventually periodic when reduced modulo an integer $M > 1$. Corollary 5.7 allows us to bound the corresponding periods for $R_k(n)$ modulo M. Namely, let Q be the associated quasi-period (by Corollary 5.7 we can always choose $Q = \mathrm{lcm}(1, 2, \ldots, k)$; as well as $Q = 3$ if $k = 3$). Then we find

$$R_k(n + MQ) \equiv R_k(n) \pmod{M}$$

for all $n > k^2$, provided that M is such that, for all $r \in \{0, 1, \ldots, Q-1\}$, the values $R_k(mQ + r)$ are produced by polynomials in m with coefficients whose denominators are coprime to M (these polynomials necessarily have rational coefficients because they are integer-valued).

Example 6.1. For instance, in the case $k = 3$, we find that $R_3(n + 3M) \equiv R_3(n)$ modulo M for all $n > 9$. Likewise, $R_4(n + 12M) \equiv R_4(n)$ modulo M for all $n > 16$ as well as $R_5(n + 60M) \equiv R_5(n)$ modulo M for all $n > 25$. In the cases $k = 3$ and $k = 4$, these observations are also made in [Sha25]. We note that in the cases $k \le 5$ no restriction on M is needed. On the other hand, for instance for $k = 6$, the congruences $R_6(n+60M) \equiv R_6(n)$ modulo M only hold for all $n > 36$ provided that M is coprime to 3.

Remarkably, much stronger congruences appear to hold in the case $M = 2$. As observed in [Sha25], for $n > k^2$, the sequences $R_k(n)$ modulo 2 are periodic with period 2 if $k \in \{2, 3\}$ and period 8 if $k \in \{4, 5\}$. Likewise, by explicitly computing the quasi-polynomial representation in Corollary 5.7, we find that the corresponding period is 24 if $k \in \{6, 7\}$, 48 if $k \in \{8, 9\}$, and 480 if $k \in \{10, 11\}$. It is natural to wonder whether the sequences $R_{2k}(n)$ and $R_{2k+1}(n)$ modulo 2 always have the same (eventual) period. We leave this question for future work. More generally, it would be of interest to investigate further congruential properties of the numbers $R_k(n)$, such as analogs of the Ramanujan congruences for the partition function. Such investigations have been initiated and developed by Kronholm [Kro05], [Kro13] for the related numbers $p(n, k)$ of partitions of n into exactly k parts. For recent subsequent work in this direction, we refer to [EKL23] and the references therein.

Finally, we wonder if it is fruitful to investigate the bivariate generating function

$$\boldsymbol{F}(q, z) = \sum_{k=0}^{\infty} F_k(q) z^k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f_k(n) q^n z^k$$

in order to increase our understanding of the underlying numbers $R_k(n) = f_k(n - T_k)$. In this direction, we note that, if we similarly write

$$\boldsymbol{A}(q, z) = \sum_{d=0}^{\infty} A_d(q) z^d,$$

then Lemma 2.1 becomes equivalent to the simple relationship

$$\boldsymbol{F}(q, z) = \boldsymbol{A}(q, qz) \boldsymbol{A}(q, z).$$

References

[AE04] George E. Andrews and Kimmo Eriksson. *Integer Partitions*. Cambridge University Press, 2004.

[And76] George E. Andrews. *The Theory of Partitions*. Addison-Wesley, 1976.

[And03] George E. Andrews. Partitions: At the interface of q-series and modular forms. *The Ramanujan Journal*, 7(1):385–400, March 2003.

[CS16] Yuriy Choliy and Andrew V. Sills. A formula for the partition function that "counts". *Annals of Combinatorics*, 20:301–316, 2016.

[DG14] Michael Drmota and Stefan Gerhold. Disproof of a conjecture by Rademacher on partial fractions. *Proceedings of the American Mathematical Society, Series B*, 1(11):121–134, 2014.

[EKL23] Dennis Eichhorn, Brandt Kronholm, and Acadia Larsen. Cranks for partitions with bounded largest part. *Proceedings of the American Mathematical Society*, 151(8):3291–3303, August 2023.

[EvdPSW03] Graham Everest, Alf van der Poorten, Igor Shparlinski, and Thomas Ward. *Recurrence Sequences*, volume 104 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[KP11] Manuel Kauers and Peter Paule. *The Concrete Tetrahedron*. Springer-Verlag, Wien, 2011.

[Kro05] Brandt Kronholm. On congruence properties of p(n, m). *American Mathematical Society*, 133(10):2891–2895, 2005.

[Kro13] Brandt Kronholm. Generalized congruence properties of the restricted partition function $p(n, m)$. *The Ramanujan Journal*, 30(3):425–436, April 2013.

[Mun07] Augustine O. Munagi. Computation of q-partial fractions. *Integers*, 7(1):#A25, 21 p., 2007.

[Mun08] Augustine O. Munagi. The Rademacher conjecture and q-partial fractions. *The Ramanujan Journal*, 15(3):339–347, April 2008.

[OEI25] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. https://oeis.org/, 2025. Published electronically.

[O'S16] Cormac O'Sullivan. Asymptotics for the partial fractions of the restricted partition generating function I. *International Journal of Number Theory*, 12(6):1421–1474, September 2016.

[PS78] George Pólya and Gabor Szegő. *Problems and Theorems in Analysis I: Series, Integral Calculus, Theory of Functions*. Springer, 1978.

[RA05] Jorge L. Ramírez Alfonsín. *The Diophantine Frobenius Problem*. Oxford University Press, Oxford, 2005.

[Rad73] Hans Rademacher. *Topics in Analytic Number Theory*, volume 169 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1973.

[Sha25] N. Guru Sharan. Rook decomposition of the partition function. *Preprint*, 2025. Available to reviewers at: https://drive.google.com/file/d/1TPJbS9DrR5cX3SjB_2VQMDIOxDUK16Rm.

[Sil19] Andrew V. Sills. The combinatorics of MacMahon's partial fractions. *Annals of Combinatorics*, 23(3):1073–1086, November 2019.

[Sta97] Richard P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, 1997.

[SZ12] Andrew V. Sills and Doron Zeilberger. Formulæ for the number of partitions of n into at most m parts (using the quasi-polynomial ansatz). *Advances in Applied Mathematics*, 48(5):640–645, May 2012.

[SZ13] Andrew V. Sills and Doron Zeilberger. Rademacher's infinite partial fraction conjecture is (almost certainly) false. *Journal of Difference Equations and Applications*, 19(4):680–689, April 2013.