LOG-SINE EVALUATIONS OF MAHLER MEASURES, II

This paper is dedicated to the memory of John Selfridge

David Borwein
Department of Mathematics, University of Western Ontario, London, ONT, N6A 5B7, Canada
dborwein@uwo.ca

Jonathan M. Borwein[1]
Centre for Computer-assisted Research Mathematics and its Applications (CARMA), University of Newcastle, Callaghan, NSW 2308, Australia
jonathan.borwein@newcastle.edu.au

Armin Straub
Department of Mathematics, Tulane University, New Orleans, LA 70118, United States
astraub@tulane.edu

James Wan
Centre for Computer-assisted Research Mathematics and its Applications (CARMA), University of Newcastle, Callaghan, NSW 2308, Australia
james.wan@newcastle.edu.au

Received: , Revised: , Accepted: , Published:

Abstract

We continue our analysis of higher and multiple Mahler measures using log-sine integrals as started in [7, 8]. This motivates a detailed study of various multiple polylogarithms [4] and worked examples are given. Our techniques enable the reduction of several multiple Mahler measures, and supply an easy proof of two conjectures by Boyd.

1. Introduction

In [7] the classical log-sine integrals and their extensions were used to develop a variety of results relating to higher and multiple Mahler measures [10, 18]. The utility of this approach was such that we continue the work herein. Among other things, it allows us to tap into a rich analytic literature [20]. In [8] the computational underpinnings of such studies are illuminated. The use of related integrals is currently being exploited for multi-zeta value studies [22]. Such evaluations are also useful for physics [17]: log-sine integrals appeared for instance in the calculation of higher terms in the ε-expansion of various Feynman diagrams [13, 16]. Of particular importance are the log-sine integrals at the special values $\pi/3$, $\pi/2$, $2\pi/3$, and π. The log-sine integrals also come up in many

[1] Supported in part by the Australian Research Council and the University of Newcastle

settings in number theory and analysis: classes of inverse binomial sums can be expressed in terms of generalized log-sine integrals [5, 12].

The structure of this article is as follows. In Section 2 our basic tools are described. After providing necessary results on log-sine integrals in Section 3, we turn to relationships between random walks and Mahler measures in Section 4. In particular, we will be interested in the multiple Mahler measure $\mu_n(1+x+y)$ which has a fine hypergeometric generating function (23) and a natural trigonometric representation (25) as a double integral.

In Section 5 we directly expand (23) and use known results from the ε-expansion of hypergeometric functions [11, 12] to obtain $\mu_n(1+x+y)$ in terms of multiple inverse binomial sums. In the cases $n = 1, 2, 3$ this leads to explicit polylogarithmic evaluations.

An alternative approach based of the double integral representation (25) is taken up in Section 6 which considers the evaluation of the inner integral in (25). Aided by combinatorics, we show in Theorems 6.3 and 6.12 that these can always be expressed in terms of multiple harmonic polylogarithms of weight k. Accordingly, we demonstrate in Section 6.3 how these polylogarithms can be reduced explicitly for low weights. In Section 7.1 we reprise from [7] the evaluation of $\mu_2(1+x+y)$. Then in Section 7.2 we apply our general results from Section 6 to a conjectural evaluation of $\mu_3(1+x+y)$.

In Section 8 we finish with an elementary proof of two recently established 1998 conjectures of Boyd and use these tools to obtain a new Mahler measure.

2. Preliminaries

For k functions (typically Laurent polynomials) in n variables the *multiple Mahler measure*, introduced in [18], is defined as

$$\mu(P_1, P_2, \ldots, P_k) := \int_0^1 \cdots \int_0^1 \prod_{j=1}^k \log \left| P_j \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n} \right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n.$$

When $P = P_1 = P_2 = \cdots = P_k$ this devolves to a *higher Mahler measure*, $\mu_k(P)$, as introduced and examined in [18]. When $k = 1$ both reduce to the standard (logarithmic) *Mahler measure* [10].

We also recall *Jensen's formula*:

$$\int_0^1 \log \left| \alpha - e^{2\pi i\, t} \right| \,\mathrm{d}t = \log \left(|\alpha| \vee 1 \right), \tag{1}$$

where $x \vee y = \max(x, y)$. An easy consequence of Jensen's formula is that for complex constants a and b we have

$$\mu(ax + b) = \log |a| \vee \log |b|. \tag{2}$$

In the following development,

$$\mathrm{Li}_{a_1,\ldots,a_k}(z) := \sum_{n_1 > \cdots > n_k > 0} \frac{z^{n_1}}{n_1^{a_1} \cdots n_k^{a_k}}$$

denotes the *generalized polylogarithm* as is studied, for instance, in [5] and in [1, Ch. 3]. For example, $\mathrm{Li}_{2,1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \sum_{j=1}^{k-1} \frac{1}{j}$. In particular, $\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k}$ is the *polylogarithm of order* k and

$$\mathrm{Ti}_k\,(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^k}$$

is the related *inverse tangent of order* k. We use the same notation for the analytic continuations of these functions.

Moreover, *multiple zeta values* are denoted by

$$\zeta(a_1, \ldots, a_k) := \mathrm{Li}_{a_1,\ldots,a_k}(1).$$

Similarly, we consider the *multiple Clausen functions* (Cl) and *multiple Glaisher functions* (Gl) of depth k which are given by

$$\mathrm{Cl}_{a_1,\ldots,a_k}\,(\theta) = \left\{ \begin{array}{ll} \mathrm{Im}\ \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ even} \\ \mathrm{Re}\ \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ odd} \end{array} \right\}, \tag{3}$$

$$\mathrm{Gl}_{a_1,\ldots,a_k}\,(\theta) = \left\{ \begin{array}{ll} \mathrm{Re}\ \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ even} \\ \mathrm{Im}\ \mathrm{Li}_{a_1,\ldots,a_k}(e^{i\theta}) & \text{if } w \text{ odd} \end{array} \right\}, \tag{4}$$

where $w = a_1 + \ldots + a_k$ is the weight of the function. As illustrated in (7), the Clausen and Glaisher functions alternate between being cosine and sine series with the parity of the dimension. Of particular importance will be the case of $\theta = \pi/3$ which has also been considered in [5].

Our other notation and usage is largely consistent with that in [20] and the newly published [21], in which most of the requisite material is described. Finally, a recent elaboration of what is meant when we speak about evaluations and "closed forms" is to be found in [6].

3. Log-sine integrals

For $n = 1, 2, \ldots$, we consider the *log-sine integrals* defined by

$$\mathrm{Ls}_n\,(\sigma) := -\int_0^{\sigma} \log^{n-1}\left|2\,\sin\frac{\theta}{2}\right|\,\mathrm{d}\theta \tag{5}$$

and, for $k = 0, 1, \ldots, n-1$, their generalized versions

$$\mathrm{Ls}_n^{(k)}\,(\sigma) := -\int_0^{\sigma} \theta^k\,\log^{n-1-k}\left|2\,\sin\frac{\theta}{2}\right|\,\mathrm{d}\theta. \tag{6}$$

This is the notation used by Lewin [19, 20]. In each case the modulus is not needed for $0 \leq \sigma \leq 2\pi$.

We observe that $\mathrm{Ls}_1\,(\sigma) = -\sigma$ and that $\mathrm{Ls}_n^{(0)}\,(\sigma) = \mathrm{Ls}_n\,(\sigma)$. In particular,

$$\mathrm{Ls}_2\,(\sigma) = \mathrm{Cl}_2\,(\sigma) := \sum_{n=1}^{\infty} \frac{\sin(n\sigma)}{n^2} \tag{7}$$

is the Clausen function introduced in (3). Various log-sine integral evaluations may be found in [20, §7.6 & §7.9].

3.1. Log-sine integrals at π

We first recall that the log-sine integrals at π can always be evaluated in terms of zeta values. This is a consequence of the exponential generating function [20, Eqn. (7.109)]

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}\mathrm{Ls}_{m+1}(\pi)\,\frac{u^m}{m!}=\frac{\Gamma(1+u)}{\Gamma^2\left(1+\frac{u}{2}\right)}=\binom{u}{u/2}. \tag{8}$$

This will be revisited and put in context in Section 4. Here we only remark that, by the very definition, log-sine integrals at π correspond to very basic multiple Mahler measures:

$$\mu_m(1+x)=-\frac{1}{\pi}\,\mathrm{Ls}_{m+1}(\pi) \tag{9}$$

Example 3.1 (Values of $\mathrm{Ls}_n(\pi)$)**.** For instance, we have $\mathrm{Ls}_2(\pi)=0$ as well as

$$\begin{aligned}
-\mathrm{Ls}_3(\pi)&=\frac{1}{12}\pi^3\\
\mathrm{Ls}_4(\pi)&=\frac{3}{2}\pi\,\zeta(3)\\
-\mathrm{Ls}_5(\pi)&=\frac{19}{240}\pi^5\\
\mathrm{Ls}_6(\pi)&=\frac{45}{2}\pi\,\zeta(5)+\frac{5}{4}\pi^3\zeta(3)\\
-\mathrm{Ls}_7(\pi)&=\frac{275}{1344}\pi^7+\frac{45}{2}\pi\,\zeta^2(3)\\
\mathrm{Ls}_8(\pi)&=\frac{2835}{4}\pi\,\zeta(7)+\frac{315}{8}\pi^3\zeta(5)+\frac{133}{32}\pi^5\zeta(3),
\end{aligned}$$

and so forth. Note that these values may be conveniently obtained from (8) by a computer algebra system as the following snippet of *Maple* code demonstrates:

```
for k to 6 do simplify(subs(x=0,diff(Pi*binomial(x,x/2),x$k))) od;
```

More general log-sine evaluations with an emphasis on automatic evaluations have been studied in [8]. ◇

For general log-sine integrals, the following computationally effective exponential generating function was obtained in [8].

Theorem 3.2 (Generating function for $\mathrm{Ls}^{(k)}_{n+k+1}(\pi)$)**.** *For $2|\mu|<\lambda<1$ we have*

$$\sum_{n,k\geq 0}\mathrm{Ls}^{(k)}_{n+k+1}(\pi)\,\frac{\lambda^n}{n!}\frac{(i\mu)^k}{k!}=-i\sum_{n\geq 0}\binom{\lambda}{n}\frac{(-1)^n\,\mathrm{e}^{i\pi\frac{\lambda}{2}}-\mathrm{e}^{i\pi\mu}}{\mu-\frac{\lambda}{2}+n}. \tag{10}$$

One may extract one-variable generating functions from (10). For instance,

$$\sum_{n=0}^{\infty}\mathrm{Ls}^{(1)}_{n+2}(\pi)\,\frac{\lambda^n}{n!}=\sum_{n=0}^{\infty}\binom{\lambda}{n}\frac{-1+(-1)^n\cos\frac{\pi\lambda}{2}}{\left(n-\frac{\lambda}{2}\right)^2}.$$

The log-sine integrals at $\pi/3$ are especially useful as illustrated in [5] and are discussed at some length in [7] where other applications to Mahler measures are given.

3.2. Extensions of the log-sine integrals

It is possible to extend some of these considerations to the log-sine-cosine integrals

$$\mathrm{Lsc}_{m,n}\left(\sigma\right) := -\int_0^\sigma \log^{m-1}\left|2\,\sin\frac{\theta}{2}\right|\,\log^{n-1}\left|2\,\cos\frac{\theta}{2}\right|\,\mathrm{d}\theta. \tag{11}$$

Then $\mathrm{Lsc}_{m,1}\left(\sigma\right) = \mathrm{Ls}_m\left(\sigma\right)$ and $\mathrm{Lsc}_{m,n}\left(\sigma\right) = \mathrm{Lsc}_{n,m}\left(\sigma\right)$. As in (9), these are related to basic multiple Mahler measures. Namely, if we set

$$\mu_{m,n}(1-x,1+x) := \mu(\underbrace{1-x,\cdots,1-x}_{m},\underbrace{1+x,\cdots,1+x}_{n}) \tag{12}$$

then, immediately from the definition, we obtain the following:

Theorem 3.3 (Evaluation of $\mu_{m,n}(1-x,1+x)$)**.** *For non-negative integers m, n,*

$$\mu_{m,n}(1-x,1+x) = -\frac{1}{\pi}\,\mathrm{Lsc}_{m+1,n+1}\left(\pi\right). \tag{13}$$

In every case this is evaluable in terms of zeta values. Indeed, using the result in [20, §7.9.2, (7.114)], we obtain the generating function

$$\mathrm{gs}(u,v) := -\frac{1}{\pi}\sum_{m,n=0}^{\infty}\mathrm{Lsc}_{m+1,n+1}\left(\pi\right)\,\frac{u^m}{m!}\,\frac{v^n}{n!} = \frac{2^{u+v}}{\pi}\,\frac{\Gamma\left(\frac{1+u}{2}\right)\Gamma\left(\frac{1+v}{2}\right)}{\Gamma\left(1+\frac{u+v}{2}\right)}. \tag{14}$$

From the duplication formula for the gamma function this can be rewritten as

$$\mathrm{gs}(u,v) = \binom{u}{u/2}\binom{v}{v/2}\frac{\Gamma\left(1+\frac{u}{2}\right)\Gamma\left(1+\frac{v}{2}\right)}{\Gamma\left(1+\frac{u+v}{2}\right)},$$

so that

$$\mathrm{gs}(u,0) = \binom{u}{u/2} = \mathrm{gs}(u,u).$$

From here it is apparent that (14) is an extension of (8):

Example 3.4 (Values of $\mathrm{Lsc}_{n,m}\left(\pi\right)$)**.** For instance,

$$\begin{aligned}
\mu_{2,1}(1-x,1+x) &= \mu_{1,2}(1-x,1+x) = \frac{1}{4}\zeta(3),\\
\mu_{3,2}(1-x,1+x) &= \frac{3}{4}\zeta(5) - \frac{1}{8}\pi^2\zeta(3),\\
\mu_{6,3}(1-x,1+x) &= \frac{315}{4}\zeta(9) + \frac{135}{32}\pi^2\zeta(7) + \frac{309}{128}\pi^4\zeta(5) - \frac{45}{256}\pi^6\zeta(3) - \frac{1575}{32}\zeta^3(3).
\end{aligned}$$

As in Example 3.1 this can be easily obtained with a line of code in a computer algebra system such as *Mathematica* or *Maple*. ◇

Remark 3.5. From $\mathrm{gs}(u,-u)=\sec(\pi u/2)$ we may deduce that, for $n=0,1,2,\ldots$,

$$\sum_{k=0}^{n}(-1)^k\mu_{k,n-k}(1-x,1+x)=|E_{2n}|\,\frac{\left(\frac{\pi}{2}\right)^{2n}}{(2n)!}=\frac{4}{\pi}\,L_{-4}(2n+1),$$

where E_{2n} are the even Euler numbers: $1,-1,5,-61,1385\ldots.$ ◇

A more recondite *extended log-sine integral of order three* is developed in [20, §8.4.3] from properties of the trilogarithm. It is defined by

$$\mathrm{Ls}_3\left(\theta,\omega\right):=-\int_0^\theta \log\left|2\sin\frac{\sigma}{2}\right|\,\log\left|2\sin\frac{\sigma+\omega}{2}\right|\,\mathrm{d}\sigma, \tag{15}$$

so that $\mathrm{Ls}_3\left(\theta,0\right)=\mathrm{Ls}_3\left(\theta\right)$. This extended log-sine integral reduces as follows:

$$\begin{aligned}-\mathrm{Ls}_3\left(2\theta,2\omega\right)&=\frac{1}{2}\,\mathrm{Ls}_3\left(2\omega\right)-\frac{1}{2}\,\mathrm{Ls}_3\left(2\theta\right)-\frac{1}{2}\,\mathrm{Ls}_3\left(2\theta+2\omega\right)\\&\quad-2\,\mathrm{Im}\,\mathrm{Li}_3\left(\frac{\sin(\theta)\mathrm{e}^{i\omega}}{\sin(\theta+\omega)}\right)+\theta\log^2\left(\frac{\sin(\theta)}{\sin(\theta+\omega)}\right)\\&\quad+\log\left(\frac{\sin(\theta)}{\sin(\theta+\omega)}\right)\left\{\mathrm{Cl}_2\left(2\theta\right)+\mathrm{Cl}_2\left(2\omega\right)-\mathrm{Cl}_2\left(2\theta+2\omega\right)\right\}.\end{aligned} \tag{16}$$

We note that $-\frac{1}{2\pi}\,\mathrm{Ls}_3\left(2\pi,\omega\right)=\mu(1-x,1-e^{i\omega}x)$ but this is more easily evaluated by Fourier techniques. Indeed one has:

Proposition 3.6 (A dilogarithmic measure, part I [18])**.** *For two complex numbers u and v we have*

$$\mu(1-u\,x,1-v\,x)=\begin{cases}\frac{1}{2}\ \mathrm{Re}\ \mathrm{Li}_2\left(v\overline{u}\right), & \text{if } |u|\le 1, |v|\le 1,\\ \frac{1}{2}\ \mathrm{Re}\ \mathrm{Li}_2\left(\frac{v}{\overline{u}}\right), & \text{if } |u|\ge 1, |v|\le 1,\\ \frac{1}{2}\ \mathrm{Re}\ \mathrm{Li}_2\left(\frac{1}{v\overline{u}}\right)+\log|u|\log|v|, & \text{if } |u|\ge 1, |v|\ge 1.\end{cases} \tag{17}$$

This is proven much as is (77) of Proposition 7.2. In Lewin's terms [20, A.2.5] for $0<\theta\le 2\pi$ and $r\ge 0$ we may write

$$\mathrm{Re}\ \mathrm{Li}_2\left(re^{i\theta}\right)=:\mathrm{Li}_2\left(r,\theta\right)=-\frac{1}{2}\int_0^r\log\left(t^2+1-2t\cos\theta\right)\frac{\mathrm{d}t}{t}, \tag{18}$$

with the reflection formula

$$\mathrm{Li}_2\left(r,\theta\right)+\mathrm{Li}_2\left(\frac{1}{r},\theta\right)=\zeta(2)-\frac{1}{2}\,\log^2 r+\frac{1}{2}\,(\pi-\theta)^2. \tag{19}$$

This leads to:

Proposition 3.7 (A dilogarithmic measure, part II)**.** *For complex numbers $u=re^{i\theta}$ and $v=se^{i\tau}$ we have*

$$\mu(1-u\,x,1-v\,x)=\begin{cases}\frac{1}{2}\ \mathrm{Li}_2\left(rs,\theta-\tau\right) & \text{if } r\le 1, s\le 1,\\ \frac{1}{2}\ \mathrm{Li}_2\left(\frac{s}{r},\theta+\tau\right), & \text{if } r\ge 1, s\le 1,\\ \frac{1}{2}\ \mathrm{Li}_2\left(\frac{1}{sr},\theta-\tau\right)+\log r\log s, & \text{if } r\ge 1, s\ge 1.\end{cases} \tag{20}$$

We remark that Proposition 3.7 and equation (19) allow for efficient numerical computation.

4. Mahler measures and moments of random walks

The s-th moments of an n-step uniform random walk are given by

$$W_n(s) = \int_0^1 \cdots \int_0^1 \left| \sum_{k=1}^n e^{2\pi i t_k} \right|^s \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

and their relation with Mahler measure is observed in [9]. In particular,

$$W_n'(0) = \mu(1 + x_1 + \ldots + x_{n-1}),$$

with the cases $2 \le n \le 6$ discussed in [7].

Higher derivatives of W_n correspond to higher Mahler measures:

$$W_n^{(m)}(0) = \mu_m(1 + x_1 + \ldots + x_{n-1}). \tag{21}$$

The evaluation $W_2(s) = \binom{s}{s/2}$ thus explains and proves the generating function (8); in other words, we find that

$$W_2^{(m)}(0) = -\frac{1}{\pi} \operatorname{Ls}_{m+1}(\pi). \tag{22}$$

As a consequence of the study of random walks in [9] we record the following generating function for $\mu_m(1 + x + y)$ which follows from (21) and the hypergeometric expression for W_3 in [9]. There is a corresponding expression for W_4, the generating function of $\mu_m(1 + x + y + z)$, in terms of a single Meijer-G function [9].

Theorem 4.1 (Hypergeometric form for $W_3(s)$)**.** *For complex $|s| < 2$, we may write*

$$W_3(s) = \sum_{n=0}^{\infty} \mu_n(1 + x + y) \frac{s^n}{n!} = \frac{\sqrt{3}}{2\pi} 3^{s+1} \frac{\Gamma(1 + \frac{s}{2})^2}{\Gamma(s+2)} \, {}_3F_2 \left(\left. {\frac{s+2}{2}, \frac{s+2}{2}, \frac{s+2}{2} \atop 1, \frac{s+3}{2}} \right| \frac{1}{4} \right) \tag{23}$$

$$= \frac{\sqrt{3}}{\pi} \left(\frac{3}{2} \right)^{s+1} \int_0^1 \frac{z^{1+s} {}_2F_1 \left(\left. {1+\frac{s}{2}, 1+\frac{s}{2} \atop 1} \right| \frac{z^2}{4} \right)}{\sqrt{1 - z^2}} \, \mathrm{d}z. \tag{24}$$

Proof. Equation (23) is proven in [9], while (24) is a consequence of (23) and [21, Eqn. (16.5.2)]. □

We shall exploit Theorem 4.1 next, in Section 5. For integers $n \ge 1$ we also have

$$\mu_n(1 + x + y) = \frac{1}{4\pi^2} \int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \left(\operatorname{Re} \log\left(1 - 2\sin(\theta) \mathrm{e}^{i\,\omega}\right) \right)^n \mathrm{d}\omega, \tag{25}$$

as follows directly from the definition and some simple trigonometry, since $\operatorname{Re} \log z = \log |z|$. This is the basis for the evaluations of Section 6. In particular, in Section 6 we will evaluate the inner integral in terms of multiple harmonic polylogarithms.

5. Epsilon expansion of W_3

In this section we use known results from the ε-expansion of hypergeometric functions [11, 12] to obtain $\mu_n(1+x+y)$ in terms of multiple inverse binomial sums. We then derive complete evaluations of $\mu_1(1+x+y)$, $\mu_2(1+x+y)$ and $\mu_3(1+x+y)$. An alternative approach will be pursued in Sections 6 and 7.

In light of Theorem 4.1, the evaluation of $\mu_n(1+x+y)$ is essentially reduced to the Taylor expansion

$$ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right) = \sum_{n=0}^{\infty} \alpha_n \varepsilon^n. \tag{26} $$

Indeed, from (23) and Leibniz' rule we have

$$ \mu_n(1+x+y) = \frac{\sqrt{3}}{2\pi} \sum_{k=0}^{n} \binom{n}{k} \alpha_k \beta_{n-k} \tag{27} $$

where β_k is defined by

$$ 3^{\varepsilon+1} \frac{\Gamma(1+\frac{\varepsilon}{2})^2}{\Gamma(2+\varepsilon)} = \sum_{n=0}^{\infty} \beta_n \varepsilon^n. \tag{28} $$

Note that β_k is easy to compute as illustrated in Example 3.1. The expansion of hypergeometric functions in terms of their parameters as in (26) occurs in physics [11, 12] in the context of the evaluation of Feynman diagrams and is commonly referred to as *epsilon expansion*, thus explaining the choice of variable in (26).

Remark 5.1. From (28) we see that the β_n may be computed directly from the coefficients γ_n defined by the Taylor expansion

$$ \frac{\Gamma(1+\frac{\varepsilon}{2})^2}{\Gamma(1+\varepsilon)} = \frac{1}{\binom{\varepsilon}{\varepsilon/2}} = \sum_{n=0}^{\infty} \gamma_n \varepsilon^n. $$

Appealing to (8) we find that γ_n is recursively determined by $\gamma_0 = 1$ and

$$ \gamma_n = \frac{1}{\pi} \sum_{k=1}^{n} \mathrm{Ls}_{k+1}(\pi) \frac{\gamma_{n-k}}{k!}. $$

In particular, the results of Section 3.1 show that γ_n can always be expressed in terms of zeta values. Accordingly, β_n evaluates in terms of $\log 3$ and zeta values. ◇

Let $S_k(j) := \sum_{m=1}^{j} \frac{1}{m^k}$ denote the harmonic numbers of order k. Following [12] we abbreviate $S_k := S_k(j-1)$ and $\bar{S}_k := S_k(2j-1)$ in order to make it more clear which results in this reference contribute to the evaluations below. As in [11, Appendix B], we use the duplication formula $(2a)_{2j} = 4^j (a)_j (a+1/2)_j$ as well as the expansion

$$ \frac{(m+a\varepsilon)_j}{(m)_j} = \exp\left[-\sum_{k=1}^{\infty} \frac{(-a\varepsilon)^k}{k} \left[S_k(j+m-1) - S_k(m-1)\right]\right], \tag{29} $$

for m a positive integer, to write

$$
\begin{aligned}
{}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2}\\ 1,\frac{\varepsilon+3}{2}\end{matrix}\middle|\frac{1}{4}\right) &= \sum_{j=0}^{\infty}\frac{(1+\varepsilon/2)_j^3}{4^j(j!)^2(3/2+\varepsilon/2)_j}\\
&= \sum_{j=0}^{\infty}\frac{(1+\varepsilon/2)_j^4}{(j!)^2(2+\varepsilon)_{2j}}\\
&= \sum_{j=0}^{\infty}\frac{2}{j+1}\frac{1}{\binom{2(j+1)}{j+1}}\left[\frac{(1+\varepsilon/2)_j}{j!}\right]^4\left[\frac{(2+\varepsilon)_{2j}}{(2j+1)!}\right]^{-1}\\
&= \sum_{j=1}^{\infty}\frac{2}{j}\frac{1}{\binom{2j}{j}}\exp\left[\sum_{k=1}^{\infty}\frac{(-\varepsilon)^k}{k}A_{k,j}\right] \qquad (30)
\end{aligned}
$$

where

$$A_{k,j} := S_k(2j-1) - 1 - 4\frac{S_k(j-1)}{2^k} = \sum_{m=2}^{2j-1}\frac{2(-1)^{m+1}-1}{m^k}. \qquad (31)$$

We can now read off the terms α_n of the ε-expansion (26):

Theorem 5.2. *For $n = 0, 1, 2, \dots$*

$$\alpha_n = [\varepsilon^n]\ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2}\\ 1,\frac{\varepsilon+3}{2}\end{matrix}\middle|\frac{1}{4}\right) = (-1)^n\sum_{j=1}^{\infty}\frac{2}{j}\frac{1}{\binom{2j}{j}}\sum\prod_{k=1}^{n}\frac{A_{k,j}^{m_k}}{m_k!k^{m_k}} \qquad (32)$$

where the inner sum is over all non-negative integers $m_1, \dots, m_n$ such that $m_1+2m_2+\dots+nm_n = n$.

Proof. Equation (32) may be derived from (30) using, for instance, Faà di Bruno's formula for the n-th derivative of the composition of two functions. □

Example 5.3 (α_0, α_1 and α_2)**.** In particular,

$$
\begin{aligned}
\alpha_1 = [\varepsilon]\ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2}\\ 1,\frac{\varepsilon+3}{2}\end{matrix}\middle|\frac{1}{4}\right) &= -\sum_{j=1}^{\infty}\frac{2}{j}\frac{1}{\binom{2j}{j}}A_{1,j}\\
&= -\sum_{j=1}^{\infty}\frac{2}{j}\frac{1}{\binom{2j}{j}}\left[\bar{S}_1 - 2S_1 - 1\right].
\end{aligned}
$$

Such *multiple inverse binomial sums* are studied in [12]. In particular, using [12, (2.20), (2.21)] we find

$$\alpha_0 = \frac{2\pi}{3\sqrt{3}}, \qquad (33)$$

$$\alpha_1 = \frac{2}{3\sqrt{3}}\left[\pi - \pi\log 3 + \mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right]. \qquad (34)$$

For the second term α_2 in the ε-expansion (30) produces

$$[\varepsilon^2]\,{}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2}\\ 1,\frac{\varepsilon+3}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right)=\sum_{j=1}^{\infty}\frac{1}{j}\frac{1}{\binom{2j}{j}}\left[A_{1,j}^2+A_{2,j}\right]$$
$$=\sum_{j=1}^{\infty}\frac{1}{j}\frac{1}{\binom{2j}{j}}\left[\bar{S}_2-S_2+(\bar{S}_1-2S_1)^2-2\bar{S}_1+4S_1\right].$$

Using [12, (2.8),(2.22)–(2.24)] we obtain

$$\alpha_2=\frac{2}{3\sqrt{3}}\left[\frac{\pi}{72}-\pi\log 3+\frac{1}{2}\pi\log 3+(1-\log 3)\,\mathrm{Ls}_2\left(\frac{\pi}{3}\right)\right.$$
$$\left.+\frac{3}{2}\,\mathrm{Ls}_3\left(\frac{\pi}{3}\right)+\frac{3}{2}\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)-3\,\mathrm{Ls}_3\left(\pi\right)\right]. \tag{35}$$

◇

These results provide us with evaluations of $\mu_1(1+x+y)$ and $\mu_2(1+x+y)$ as given next. As expected, the result for $\mu_1(1+x+y)$ agrees with Smyth's original evaluation, and the result for $\mu_2(1+x+y)$ agrees with our prior evaluation in [7]. The latter evaluation will be recalled in Section 7.1.

Theorem 5.4 (Evaluation of $\mu_1(1+x+y)$ and $\mu_2(1+x+y)$)**.** *We have*

$$\mu_1(1+x+y)=\frac{1}{\pi}\,\mathrm{Ls}_2\left(\frac{\pi}{3}\right), \tag{36}$$
$$\mu_2(1+x+y)=\frac{3}{\pi}\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{\pi^2}{4}. \tag{37}$$

Proof. Using Theorem 4.1 we obtain

$$\mu_1(1+x+y)=\frac{3\sqrt{3}}{2\pi}\left[(\log 3-1)\alpha_0+\alpha_1\right]. \tag{38}$$

Combining this with equations (33) and (34) yields (36).

Again using Theorem 4.1 we find

$$\mu_2(1+x+y)=\frac{3\sqrt{3}}{2\pi}\left[(\log^2 3-2\log 3+2-\tfrac{\pi^2}{12})\alpha_0+2(\log 3-1)\alpha_1+2\alpha_2\right] \tag{39}$$

and, together with equations (33), (34) and (35), arrive at

$$\pi\mu_2(1+x+y)=3\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+3\,\mathrm{Ls}_3\left(\frac{\pi}{3}\right)-6\,\mathrm{Ls}_3\left(\pi\right)-\frac{\pi^3}{18}$$
$$=3\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{\pi^3}{4}. \tag{40}$$

The last equality follows, for instance, automatically from the results in [8]. □

Example 5.5 (α_3)**.** The evaluation of α_3 is more involved and we omit some details. Again, (30) produces

$$[\varepsilon^3]\ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2},\frac{\varepsilon+2}{2}\\ 1,\frac{\varepsilon+3}{2}\end{matrix}\middle|\frac{1}{4}\right)=-\frac{1}{3}\sum_{j=1}^{\infty}\frac{1}{j}\frac{1}{\binom{2j}{j}}\left[A_{1,j}^3+3A_{1,j}A_{2,j}+2A_{3,j}\right].$$

Using [12, (2.25)–(2.28),(2.68)–(2.70),(2.81),(2.89)] as well as results from [8] we are lead to

$$\begin{aligned}\alpha_3=\frac{2}{3\sqrt{3}}&\left[\frac{5\pi^3}{108}(1-\log 3)+\frac{1}{2}\pi\log^2 3-\frac{1}{6}\pi\log^3 3+\frac{11}{9}\pi\zeta(3)\right.\\ &+\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\left(\frac{5}{36}\pi^2-\log 3+\frac{1}{2}\log^2 3\right)-3\,\mathrm{Gl}_{2,1}\left(\frac{2\pi}{3}\right)(1-\log 3)\\ &\left.-\frac{35}{6}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)+15\,\mathrm{Cl}_{2,1,1}\left(\frac{2\pi}{3}\right)-3\,\mathrm{Lsc}_{2,3}\left(\frac{\pi}{3}\right)\right].\end{aligned}\tag{41}$$

Observe the occurrence of the log-sine-cosine integral $\mathrm{Lsc}_{2,3}\left(\frac{\pi}{3}\right)$. These integrals were defined in (11). ◇

Proceeding as in the proof of Theorem 5.4 we obtain:

Theorem 5.6 (Evaluation of $\mu_3(1+x+y)$)**.** *We have*

$$\begin{aligned}\pi\mu_3(1+x+y)=15\,\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)-18\,\mathrm{Lsc}_{2,3}\left(\frac{\pi}{3}\right)-15\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\\ -\frac{1}{4}\pi^2\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)-17\pi\zeta(3).\end{aligned}\tag{42}$$

The log-sine-cosine integral $\mathrm{Lsc}_{2,3}\left(\frac{\pi}{3}\right)$ appears to reduce further as

$$\begin{aligned}12\,\mathrm{Lsc}_{2,3}\left(\frac{\pi}{3}\right)&\overset{?[1]}{=}6\,\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)-4\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)-7\pi\zeta(3)\\ &=6\,\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)-\frac{8}{9}\mathrm{Ls}_4\left(\frac{\pi}{3}\right)-\frac{59}{9}\pi\zeta(3).\end{aligned}\tag{43}$$

This conjectural reduction also appears in [11, (A.30)] where it was found via PSLQ. Combining this with (42), we obtain an conjectural evaluation of $\mu_3(1+x+y)$ equivalent to (83).

On the other hand, it follows from [12, (2.18)] that

$$\begin{aligned}12\,\mathrm{Lsc}_{2,3}\left(\frac{\pi}{3}\right)=\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)-4\,\mathrm{Ls}_4\left(\frac{\pi}{3}\right)-\frac{1}{12}\pi\log^3 3\\ +24\,\mathrm{Ti}_4\left(\frac{1}{\sqrt{3}}\right)+12\log 3\,\mathrm{Ti}_3\left(\frac{1}{\sqrt{3}}\right)+3\log^2 3\,\mathrm{Ti}_2\left(\frac{1}{\sqrt{3}}\right).\end{aligned}\tag{44}$$

Using the known evaluations — see for instance [7, (76),(77)] — for the inverse tangent integrals of order two and three, we find that (43) is equivalent to

$$\begin{aligned}\mathrm{Ti}_4\left(\frac{1}{\sqrt{3}}\right)\overset{?[1]}{=}\frac{5}{24}\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)+\frac{7}{54}\mathrm{Ls}_4\left(\frac{\pi}{3}\right)-\frac{59}{216}\pi\zeta(3)-\frac{1}{288}\pi\log^3 3\\ -\frac{1}{2}\log 3\,\mathrm{Ti}_3\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{8}\log^2 3\,\mathrm{Ti}_2\left(\frac{1}{\sqrt{3}}\right).\end{aligned}\tag{45}$$

6. Trigonometric analysis of $\mu_n(1+x+y)$

As promised in [7] — motivated by the development outlined above — we take the analysis of $\mu_n(1+x+y)$ for $n \geq 3$ a fair distance. In light of (25) we define

$$\rho_n(\alpha) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\mathrm{Re}\,\log\left(1-\alpha\,\mathrm{e}^{i\,\omega}\right)\right)^n \,\mathrm{d}\omega \tag{46}$$

for $n \geq 0$ so that

$$\mu_n(1+x+y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho_n(|2\sin\theta|)\,\mathrm{d}\theta. \tag{47}$$

We thus typically set $\alpha = |2\sin\theta|$. Note that $\rho_0(\alpha) = 1$, $\rho_1(\alpha) = \log(|\alpha| \vee 1)$.

Proposition 6.1 (Properties of ρ_n)**.** *Let n be a positive integer.*

(a) For $|\alpha| \leq 1$ we have

$$\rho_n(\alpha) = (-1)^n \sum_{m=1}^{\infty} \frac{\alpha^m}{m^n}\,\omega_n(m), \tag{48}$$

where ω_n is defined as

$$\omega_n(m) = \sum_{\sum_{j=1}^n k_j = m} \frac{1}{2\pi}\int_{-\pi}^{\pi} \prod_{j=1}^{n} \frac{m}{k_j}\cos(k_j\omega)\,\mathrm{d}\omega. \tag{49}$$

(b) For $|\alpha| \geq 1$ we have

$$\rho_n(\alpha) = \sum_{k=0}^{n} \binom{n}{k} \log^{n-k}|\alpha|\,\rho_k\left(\frac{1}{\alpha}\right). \tag{50}$$

Proof. For (a) we use (46) to write

$$\begin{aligned}\rho_n(\alpha) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\mathrm{Re}\;\log\left(1-\alpha\mathrm{e}^{i\,\omega}\right)\right)^n \mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{-\sum_{k\geq 1}\frac{\alpha^k}{k}\,\cos(k\omega)\right\}^n \mathrm{d}\omega \\ &= (-1)^n \sum_{m=1}^{\infty}\frac{\alpha^m}{m^n}\,\omega_n(m),\end{aligned}$$

as asserted. We note that $|\omega_n(m)| \leq m^n$ and so the sum is convergent.

For (b) we now use (46) to write

$$\begin{aligned}\rho_n(\alpha) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^n\left(|\alpha|\,\left|1-\alpha^{-1}\mathrm{e}^{i\,\omega}\right|\right)\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\log|\alpha| + \log\left|1-\alpha^{-1}\mathrm{e}^{i\,\omega}\right|\right)^n\,\mathrm{d}\omega \\ &= \sum_{k=0}^{n}\binom{n}{k}\log^{n-k}|\alpha|\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^k\left|1-\alpha^{-1}\mathrm{e}^{i\,\omega}\right|\,\mathrm{d}\omega \\ &= \sum_{k=0}^{n}\binom{n}{k}\log^{n-k}|\alpha|\,\rho_k\left(\frac{1}{\alpha}\right),\end{aligned}$$

as required. □

Example 6.2 (Evaluation of ω_n and ρ_n for $n \le 2$)**.** We have $\omega_0(m) = 0$, $\omega_1(m) = \delta_0(m)$, and

$$\omega_2(0) = 1, \quad \omega_2(2m) = 2, \quad \omega_2(2m+1) = 0. \tag{51}$$

Likewise, $\rho_0(\alpha) = 1$, $\rho_1(\alpha) = \log(|\alpha| \vee 1)$, and

$$\rho_2(\alpha) = \begin{cases} \frac{1}{2}\operatorname{Li}_2(\alpha^2) & \text{for } |\alpha| \le 1, \\ \frac{1}{2}\operatorname{Li}_2\left(\frac{1}{\alpha^2}\right) + \log^2|\alpha| & \text{for } |\alpha| \ge 1, \end{cases} \tag{52}$$

where the latter follows from (51) and Proposition 6.1. ◇

We have arrived at the following description of $\mu_n(1+x+y)$:

Theorem 6.3 (Evaluation of $\mu_n(1+x+y)$)**.** *Let n be a positive integer. Then*

$$\begin{aligned}\mu_n(1+x+y) &= \frac{1}{\pi}\left\{\operatorname{Ls}_{n+1}\left(\frac{\pi}{3}\right) - \operatorname{Ls}_{n+1}(\pi)\right\} + \frac{2}{\pi}\int_0^{\pi/6}\rho_n(2\sin\theta)\,\mathrm{d}\theta \\ &\quad + \frac{2}{\pi}\sum_{k=2}^{n}\binom{n}{k}\int_{\pi/6}^{\pi/2}\log^{n-k}(2\sin\theta)\,\rho_k\left(\frac{1}{2\sin\theta}\right)\mathrm{d}\theta.\end{aligned} \tag{53}$$

Proof. Since $|\alpha| < 1$ exactly when $|\theta| < \pi/6$ we start with (47) to get

$$\begin{aligned}\mu_n(1+x+y) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho_n(|2\sin\theta|)\,\mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^{\pi/6}\rho_n(2\sin\theta)\,\mathrm{d}\theta + \frac{2}{\pi}\int_{\pi/6}^{\pi/2}\rho_n(2\sin\theta)\,\mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^{\pi/6}\rho_n(2\sin\theta)\,\mathrm{d}\theta \\ &\quad + \sum_{k=0}^{n}\binom{n}{k}\frac{2}{\pi}\int_{\pi/6}^{\pi/2}\log^{n-k}(2\sin\theta)\,\rho_k\left(\frac{1}{2\sin\theta}\right)\,\mathrm{d}\theta.\end{aligned}$$

We observe that for $k = 1$ the contribution is zero since ρ_1 is zero for $|\alpha| < 1$. After evaluating the term with $k = 0$ we arrive at (53). □

As is shown in [7],

$$\frac{1}{\pi}\left\{\mathrm{Ls}_{n+1}\left(\frac{\pi}{3}\right)-\mathrm{Ls}_{n+1}\left(\pi\right)\right\}=\mu(1+x+y_1,1+x+y_2,\ldots,1+x+y_n)$$

is a multiple Mahler measure. While log-sine integrals at π were the subject of Example 3.1 we record the following for values at $\pi/3$:

Example 6.4 (Values of $\mathrm{Ls}_n\left(\pi/3\right)$)**.** The following evaluations may be obtained with the help of the implementation[2] accompanying [8].

$$\begin{aligned}
\mathrm{Ls}_2\left(\frac{\pi}{3}\right) &= \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\\
-\mathrm{Ls}_3\left(\frac{\pi}{3}\right) &= \frac{7}{108}\,\pi^3\\
\mathrm{Ls}_4\left(\frac{\pi}{3}\right) &= \frac{1}{2}\pi\,\zeta(3)+\frac{9}{2}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\\
-\mathrm{Ls}_5\left(\frac{\pi}{3}\right) &= \frac{1543}{19440}\pi^5-6\,\mathrm{Gl}_{4,1}\left(\frac{\pi}{3}\right)\\
\mathrm{Ls}_6\left(\frac{\pi}{3}\right) &= \frac{15}{2}\pi\,\zeta(5)+\frac{35}{36}\,\pi^3\zeta(3)+\frac{135}{2}\,\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\\
-\mathrm{Ls}_7\left(\frac{\pi}{3}\right) &= \frac{74369}{326592}\pi^7+\frac{15}{2}\,\pi\zeta(3)^2-135\,\mathrm{Gl}_{6,1}\left(\frac{\pi}{3}\right)\\
\mathrm{Ls}_8\left(\frac{\pi}{3}\right) &= \frac{13181}{2592}\pi^5\zeta(3)+\frac{1225}{24}\pi^3\zeta(5)+\frac{319445}{864}\pi\zeta(7)\\
&\quad+\frac{35}{2}\pi^2\,\mathrm{Cl}_6\left(\frac{\pi}{3}\right)+\frac{945}{4}\,\mathrm{Cl}_8\left(\frac{\pi}{3}\right)+315\,\mathrm{Cl}_{6,1,1}\left(\frac{\pi}{3}\right)
\end{aligned}$$

◇

6.1. Further evaluation of ρ_n

To make further progress, we need first to determine ρ_n for $n \geq 3$. It is instructive to explore the next few cases.

Example 6.5 (Evaluation of ω_3 and ρ_3)**.** We use

$$4\,\cos(a)\cos(b)\cos(c)=\cos(a+b+c)+\cos(a-b-c)+\cos(a-b+c)+\cos(a-c+b)$$

and so derive

$$\omega_3(m)=\frac{1}{4}\sum\left\{\frac{m^3}{ijk}\colon i\pm j\pm k=0, i+j+k=m\right\}.$$

Note that we must have exactly one of $i=j+k, j=k+i$ or $k=i+j$. We thus learn that

[2] Packages are available for download from http://arminstraub.com/pub/log-sine-integrals

$\omega_3(2m+1) = 0$. Moreover, by symmetry,

$$\begin{aligned}\omega_3(2m) &= \frac{3}{4} \sum_{j+k=m} \frac{(2m)^3}{jk(j+k)} \\ &= 6 \sum_{j+k=m} \frac{m^2}{jk} = 12\, m \sum_{k=1}^{m-1} \frac{1}{k}. \end{aligned} \tag{54}$$

Hence, by Proposition 6.1,

$$\rho_3(\alpha) = -\frac{3}{2} \sum_{m=1}^{\infty} \frac{\sum_{k=1}^{m-1} \frac{1}{k}}{m^2}\, \alpha^{2m} = -\frac{3}{2}\, \mathrm{Li}_{2,1}(\alpha^2) \tag{55}$$

for $|\alpha| < 1$. ◇

6.2. A general formula for ρ_n

In the general case we have

$$\prod_{j=1}^{n} \cos(x_j) = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \cos\left(\sum_{j=1}^{n} \varepsilon_j x_j \right) \tag{56}$$

which follows inductively from $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$.

Proposition 6.6. *For integers $n, m \geq 0$ we have $\omega_n(2m+1) = 0$.*

Proof. In light of (56) the summand corresponding to the indices $k_1, \ldots, k_n$ in (49) for $\omega_n(2m+1) = 0$ is nonzero if and only if there exists $\varepsilon \in \{-1,1\}^n$ such that $\varepsilon_1 k_1 + \ldots + \varepsilon_n k_n = 0$. In other words, there is a set $S \subset \{1, \ldots, n\}$ such that

$$\sum_{j \in S} k_j = \sum_{j \notin S} k_j.$$

Thus $k_1 + \ldots + k_n = 2 \sum_{j \in S} k_j$ which contradicts $k_1 + \ldots + k_n = 2m+1$. □

Example 6.7 (Evaluation of ω_4 and ρ_4)**.** Proceeding as in Example 6.5 and employing (56), we find

$$\begin{aligned}\omega_4(2m) &= \frac{3}{8} \sum_{\substack{i+j=m \\ k+\ell=m}} \frac{(2m)^4}{ijk\ell} + \frac{1}{2} \sum_{i+j+k=m} \frac{(2m)^4}{ijk\ell} \\ &= 24m^2 \sum_{\substack{i<m \\ j<m}} \frac{1}{ij} + 24m^2 \sum_{i+j<m} \frac{1}{ij} \\ &= 48m^2 \sum_{i=1}^{m-1} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j} + 24m^2 \sum_{i=1}^{m-1} \frac{1}{i^2} + 48m^2 \sum_{i=1}^{m-1} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j}. \end{aligned} \tag{57}$$

Consequently, for $|\alpha| < 1$ and appealing to Proposition 6.1,

$$\rho_4(\alpha) = \sum_{m=1}^{\infty} \frac{\alpha^{2m}}{(2m)^4}\,\omega_4(2m) = 6\ \mathrm{Li}_{2,1,1}(\alpha^2) + \frac{3}{2}\ \mathrm{Li}_{2,2}(\alpha^2). \tag{58}$$

This suggests that $\rho_n(\alpha)$ is generally expressible as a sum of polylogarithmic terms, as will be shown next. $\diamond$

To help the general evaluation of $\omega_n(2m)$, for integers $j \geq 0$ and $m \geq 1$, let us define

$$\sigma_j(m) := \sum_{m_1+\ldots+m_j=m} \frac{1}{m_1 \cdots m_j}. \tag{59}$$

Proposition 6.8. *For positive integers n, m we have*

$$\frac{\omega_n(2m)}{m^n} = \sum_{j=1}^{n-1} \binom{n}{j} \sigma_j(m)\sigma_{n-j}(m) \tag{60}$$

where σ_j is as defined in (59).

Proof. It follows from (56) that

$$\omega_n(2m) = \sum_{k_1+\ldots+k_n=2m} \sum_{\substack{\varepsilon\in\{-1,1\}^n \\ \sum_j \varepsilon_j k_j=0}} \prod_{j=1}^{n} \frac{m}{k_j}.$$

Arguing as in Proposition 6.6 we therefore find that

$$\omega_n(2m) = \sum_{j=1}^{n-1} \binom{n}{j} \sum_{\substack{k_1+\ldots+k_j=m \\ k_{j+1}+\ldots+k_n=m}} \prod_{j=1}^{n} \frac{m}{k_j}.$$

This is equivalent to (60). $\square$

Moreover, we have a simple useful recursion:

Proposition 6.9. *Let $m \geq 1$. Then $\sigma_1(m) = 1/m$ while for $j \geq 2$ we have*

$$\sigma_j(m) = \frac{j}{m} \sum_{r=1}^{m-1} \sigma_{j-1}(r).$$

Proof. We have

$$\begin{aligned}
\sigma_j(m) &= \sum_{m_1+\ldots+m_j=m} \frac{1}{m_1 \cdots m_j} \\
&= \frac{j}{m} \sum_{m_1+\ldots+m_j=m} \frac{1}{m_1 \cdots m_{j-1}} \\
&= \frac{j}{m} \sum_{r=1}^{m-1} \sum_{m_1+\ldots+m_{j-1}=r} \frac{1}{m_1 \cdots m_{j-1}}
\end{aligned}$$

which yields the claim. $\square$

Corollary 6.10. *We have*

$$\sigma_j(m) = \frac{j!}{m} \sum_{m>m_1>\ldots>m_{j-1}>0} \frac{1}{m_1 \cdots m_{j-1}}.$$

Thus, for instance, $\sigma_2(m) = 2H_{m-1}/m$. From here, we easily re-obtain the evaluations of ω_3 and ω_4 given in Examples 6.5 and 6.7. To further illustrate Propositions 6.8 and 6.9, we now compute ρ_5 and ρ_6.

Example 6.11 (Evaluation of ρ_5 and ρ_6)**.** From Proposition 6.8,

$$\frac{\omega_5(2m)}{m^5} = 10\sigma_1(m)\sigma_4(m) + 20\sigma_2(m)\sigma_3(m).$$

Consequently, for $|\alpha| < 1$,

$$\begin{aligned}
-\rho_5(\alpha) &= \sum_{m=1}^{\infty} \frac{\alpha^{2m}}{(2m)^5}\,\omega_5(2m) \\
&= \frac{10 \cdot 4!}{32}\,\mathrm{Li}_{2,1,1,1}(\alpha^2) + \frac{20 \cdot 2! \cdot 3!}{32}\left(3\,\mathrm{Li}_{2,1,1,1}(\alpha^2) + \mathrm{Li}_{2,1,2}(\alpha^2) + \mathrm{Li}_{2,2,1}(\alpha^2)\right) \\
&= 30\,\mathrm{Li}_{2,1,1,1}(\alpha^2) + \frac{15}{2}\left(\mathrm{Li}_{2,1,2}(\alpha^2) + \mathrm{Li}_{2,2,1}(\alpha^2)\right). \qquad (61)
\end{aligned}$$

Similarly, we have for $|\alpha| < 1$,

$$\begin{aligned}
\rho_6(\alpha) = {} & 180\,\mathrm{Li}_{2,1,1,1,1}(\alpha^2) + 45\left(\mathrm{Li}_{2,1,1,2}(\alpha^2) + \mathrm{Li}_{2,1,2,1}(\alpha^2) + \mathrm{Li}_{2,2,1,1}(\alpha^2)\right) \\
& + \frac{45}{4}\,\mathrm{Li}_{2,2,2}(\alpha^2). \qquad (62)
\end{aligned}$$

From these examples the general pattern, established next, begins to transpire. ◇

In general, ρ_n evaluates as follows:

Theorem 6.12 (Evaluation of ρ_n)**.** *For $|\alpha| < 1$ and integers $n \geq 2$,*

$$\rho_n(\alpha) = \frac{(-1)^n n!}{4^n} \sum_w 4^{\ell(w)}\,\mathrm{Li}_w(\alpha^2)$$

where the sum is over all indices $w = (2, a_2, a_3, \ldots, a_{\ell(w)})$ such that $a_2, a_3, \ldots \in \{1, 2\}$ and $|w| = n$.

Proof. From Proposition 6.8 and Corollary 6.10 we have

$$\rho_n(\alpha) = \frac{(-1)^n n!}{2^n} \sum_{m=1}^{\infty} \frac{\alpha^{2m}}{m^2} \sum_{j=0}^{n-2} \sum_{\substack{m>m_1>\ldots>m_j>0 \\ m>m_{j+1}>\ldots>m_{n-2}>0}} \frac{1}{m_1 \cdots m_{n-2}}.$$

Combining the right-hand side into harmonic polylogarithms yields

$$\rho_n(\alpha) = \frac{(-1)^n n!}{2^n} \sum_{k=0}^{n-2} \sum_{\substack{a_1,\ldots,a_k \in \{1,2\} \\ a_1+\ldots+a_k=n-2}} 2^{c(a)}\,\mathrm{Li}_{2,a_1,\ldots,a_k}(\alpha^2)$$

where $c(a)$ is the number of 1s among $a_1, \ldots, a_k$. The claim follows. □

Example 6.13 (Special values of ρ_n)**.** Given Theorem 6.12, one does not expect to be able to evaluate $\rho_n(\alpha)$ explicitly at most points. Three exceptions are $\alpha = 0$ (which is trivial), $\alpha = 1$, and $\alpha = 1/\sqrt{2}$. For instance we have $\rho_4(1) = \frac{19}{240}\pi^4$ as well as $-\rho_5(1) = \frac{45}{2}\zeta(5) + \frac{5}{4}\zeta(3)$ and $\rho_6(1) = \frac{275}{1344}\pi^6 + \frac{45}{2}\zeta(3)^2$. At $\alpha = 1/\sqrt{2}$ we have

$$\rho_4\left(\frac{1}{\sqrt{2}}\right) = \frac{7}{16}\log^4 2 + \frac{3}{16}\pi^2\log^2 2 - \frac{39}{8}\zeta(3)\log 2 + \frac{13}{192}\pi^4 - 6\,\mathrm{Li}_4\left(\tfrac{1}{2}\right). \tag{63}$$

For $n \geq 5$ the expressions are expected to be more complicated. ◇

6.3. Reducing harmonic polylogarithms of low weight

Theorems 6.3 and 6.12 take us closer to a closed form for $\mu_n(1 + x + y)$. As ρ_n are expressible in terms of multiple harmonic polylogarithms of weight n, it remains to supply reductions for those of low weight. Such polylogarithms are reduced [4] by the use of the differential operators

$$(D_0 f)(x) = xf'(x) \quad \text{and} \quad (D_1 f)(x) = (1-x)f'(x)$$

depending on whether the outer index is '2' or '1'.

1. As was known to Ramanujan, and as studied further in [3, §8.1], for $0 < x < 1$,

$$\begin{aligned}\mathrm{Li}_{2,1}(x) = \frac{1}{2}\log^2(1-x)\log(x) + \log(1-x)\,\mathrm{Li}_2(1-x)\\ - \mathrm{Li}_3(1-x) + \zeta(3).\end{aligned} \tag{64}$$

   Equation (64), also given in [20], provides a useful expression numerically and symbolically. For future use, we also record the relation, obtainable as in [20, §6.4 & §6.7],

$$\begin{aligned}\mathrm{Re}\ \mathrm{Li}_{2,1}\left(\frac{1}{x}\right) + \mathrm{Li}_{2,1}(x) = \zeta(3) - \frac{1}{6}\log^3 x + \frac{1}{2}\pi^2\log x\\ - \mathrm{Li}_2(x)\log x + \mathrm{Li}_3(x) \qquad \text{for } 0 < x < 1.\end{aligned} \tag{65}$$

2. For $\mathrm{Li}_{2,2}$ we work as follows. As $(1-x)\,\mathrm{Li}'_{1,2}(x) = \mathrm{Li}_2(x)$, integration yields

$$\begin{aligned}\mathrm{Li}_{1,2}(x) = 2\,\mathrm{Li}_3(1-x) - \log(1-x)\,\mathrm{Li}_2(x) - 2\log(1-x)\,\mathrm{Li}_2(1-x)\\ - \log(1-x)^2\log(x) - 2\zeta(3).\end{aligned} \tag{66}$$

   Then, since $x\,\mathrm{Li}'_{2,2}(x) = \mathrm{Li}_{1,2}(x)$, on integrating again we obtain $\mathrm{Li}_{2,2}(x)$ in terms of polylogarithms up to order four. We appeal to various formulae in [20, §6.4.4] to arrive at

$$\begin{aligned}\mathrm{Li}_{2,2}(t) = \frac{1}{2}\log^2(1-t)\log^2 t - 2\zeta(2)\log(1-t)\log t - 2\zeta(3)\log t - \frac{1}{2}\mathrm{Li}_2^2(t)\\ + 2\,\mathrm{Li}_3(1-t)\log t - 2\int_0^t \frac{\mathrm{Li}_2(x)\log x}{1-x}\,\mathrm{d}x - \int_0^t \frac{\log(1-x)\log^2 x}{1-x}\mathrm{d}x.\end{aligned}$$

Expanding the penultimate integral as a series leads to

$$\int_0^t \frac{\mathrm{Li}_2(x)\log x}{1-x}\,\mathrm{d}x = \mathrm{Li}_{1,2}(t)\log t - \mathrm{Li}_{2,2}(t).$$

Then, using [20, A3.4 Eq. (12)] to evaluate the remaining integral, we deduce that

$$\begin{aligned}\mathrm{Li}_{2,2}(t) = &-\frac{1}{12}\log^4(1-t) + \frac{1}{3}\log^3(1-t)\log t - \zeta(2)\log^2(1-t)\\ &+ 2\log(1-t)\,\mathrm{Li}_3(t) - 2\,\zeta(3)\log(1-t) - 2\,\mathrm{Li}_4(t)\\ &- 2\,\mathrm{Li}_4\left(\frac{t}{t-1}\right) + 2\,\mathrm{Li}_4(1-t) - 2\zeta(4) + \frac{1}{2}\,\mathrm{Li}_2^2(t).\end{aligned} \tag{67}$$

3. The form for $\mathrm{Li}_{3,1}(t)$ is obtained in the same way but starting from $\mathrm{Li}_{2,1}(t)$ as given in (64). This leads to:

$$2\,\mathrm{Li}_{3,1}(t) + \mathrm{Li}_{2,2}(t) = \frac{1}{2}\,\mathrm{Li}_2^2(t). \tag{68}$$

This symmetry result, and its derivative

$$2\,\mathrm{Li}_{2,1}(t) + \mathrm{Li}_{1,2}(t) = \mathrm{Li}_1(t)\,\mathrm{Li}_2(t), \tag{69}$$

are also obtained in [25, Cor. 2 & Cor. 3] by other methods.

4. Since $\mathrm{Li}_{2,1,1}(x) = \int_0^x \mathrm{Li}_{1,1,1}(t)/t\,\mathrm{d}t$ and $\mathrm{Li}_{1,1,1}(x) = \int_0^x \mathrm{Li}_{1,1}(t)/(1-t)\,\mathrm{d}t$, we first compute $\mathrm{Li}_{1,1}(x) = \log^2(1-x)/2$ to find that $\mathrm{Li}_{1,1,1}(x) = -\log^3(1-x)/6$ (the pattern is clear). Hence

$$\begin{aligned}\mathrm{Li}_{2,1,1}(x) &= -\frac{1}{6}\int_0^x \log^3(1-t)\frac{\mathrm{d}t}{t}\\ &= \frac{\pi^4}{90} - \frac{1}{6}\log(1-t)^3\log t - \frac{1}{2}\log(1-t)^2\,\mathrm{Li}_2(1-t)\\ &\quad + \log(1-t)\,\mathrm{Li}_3(1-t) - \mathrm{Li}_4(1-t).\end{aligned} \tag{70}$$

5. In general,

$$\mathrm{Li}_{\{1\}^n}(x) = \frac{(-1)^n}{n!}\log(1-x)^n, \tag{71}$$

and therefore

$$\begin{aligned}\mathrm{Li}_{2,\{1\}^{n-1}}(x) &= \frac{(-1)^n}{n!}\int_0^x \log(1-t)^n\frac{\mathrm{d}t}{t}\\ &= \zeta(n+1) - \sum_{m=0}^{n}\frac{(-1)^{n-m}}{(n-m)!}\log(1-x)^{n-m}\,\mathrm{Li}_{m+1}(1-x).\end{aligned} \tag{72}$$

We have, inter alia, provided closed reductions for all multiple polylogarithms of weight less than five. One does not expect such complete results thereafter.

The reductions presented in this section allow us to express ρ_3 and ρ_4 in terms of polylogarithms of depth 1. Equation (64) treats ρ_3 while (58) leads to

$$\begin{aligned}\rho_4\left(\alpha^2\right) = {} & 3\left(\mathrm{Li}_3\left(\alpha^2\right) - \zeta(3) + \mathrm{Li}_3\left(1-\alpha^2\right)\right)\log\left(1-\alpha^2\right) - \frac{1}{8}\log^4\left(1-\alpha^2\right) \\ & + 3\zeta(4) - 3\,\mathrm{Li}_4\left(\frac{-\alpha^2}{1-\alpha^2}\right) - 3\,\mathrm{Li}_4\left(\alpha^2\right) - 3\,\mathrm{Li}_4\left(1-\alpha^2\right) + \frac{3}{4}\mathrm{Li}_2^2\left(1-\alpha^2\right) \\ & - \log\alpha\,\log^3\left(1-\alpha^2\right) - \left(\frac{\pi^2}{4} + 3\,\mathrm{Li}_2\left(1-\alpha^2\right)\right)\log^2\left(1-\alpha^2\right). \end{aligned} \tag{73}$$

7. Explicit evaluations of $\mu_n(1+x+y)$ for small n

We now return to the explicit evaluation of the multiple Mahler measures $\mu_k(1+x+y)$. The starting point for this section is the evaluation of $\mu_2(1+x+y)$ from [7] which is reviewed in Section 7.1 and was derived alternatively in Theorem 5.4. Building on this, we present an informal evaluation of $\mu_3(1+x+y)$ in Section 7.2. A conjectural evaluation of $\mu_4(1+x+y)$ is presented in equation (109) of the Conclusion.

7.1. Evaluation of $\mu_2(1+x+y)$

Theorem 7.1 (Evaluation of $\mu_2(1+x+y)$)**.** *We have*

$$\mu_2(1+x+y) = \frac{3}{\pi}\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{\pi^2}{4}. \tag{74}$$

By comparison, Smyth's original result may be written as (see [7])

$$\mu_1(1+x+y) = \frac{3}{2\pi}\,\mathrm{Ls}_2\left(\frac{2\pi}{3}\right) = \frac{1}{\pi}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right). \tag{75}$$

We recall from [7] that the evaluation in Theorem 7.1 is proceeded by first establishing the following dilogarithmic form.

Proposition 7.2 (A dilogarithmic representation)**.** *We have*

(a)

$$\frac{2}{\pi}\int_0^\pi \mathrm{Re}\,\mathrm{Li}_2\left(4\sin^2\theta\right)\mathrm{d}\theta = 2\zeta(2), \tag{76}$$

(b)

$$\mu_2(1+x+y) = \frac{\pi^2}{36} + \frac{2}{\pi}\int_0^{\pi/6}\mathrm{Li}_2\left(4\sin^2\theta\right)\mathrm{d}\theta. \tag{77}$$

We include the proof from [7] as it is instructive for evaluation of $\mu_3(1+x+y)$.

Proof. For (a) we define $\tau(z) := \frac{2}{\pi}\int_0^\pi \operatorname{Li}_2\left(4z\sin^2\theta\right) \mathrm{d}\theta$. This is an analytic function of z. For $|z| < 1/4$ we may use the defining series for Li_2 and expand term by term using Wallis' formula to derive

$$\begin{aligned}\tau(z) &= \frac{2}{\pi}\sum_{n\geq 1}\frac{(4z)^n}{n^2}\int_0^\pi \sin^{2n}\theta \,\mathrm{d}\theta = 4z\,{}_4F_3\left(\left.{1,1,1,\frac{3}{2} \atop 2,2,2}\right|4z\right)\\ &= 4\operatorname{Li}_2\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-4z}\right) - 2\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4z}\right)^2.\end{aligned}$$

The final equality can be obtained in *Mathematica* and then verified by differentiation. In particular, the final function provides an analytic continuation from which we obtain $\tau(1) = 2\zeta(2) + 4i\operatorname{Cl}_2\left(\frac{\pi}{3}\right)$. This yields the assertion.

For (b), commencing much as in [18, Thm. 11], we write

$$\mu_2(1+x+y) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \operatorname{Re}\log\left(1-2\sin(\theta)\mathrm{e}^{i\,\omega}\right)^2 \,\mathrm{d}\omega\,\mathrm{d}\theta.$$

We consider the inner integral $\rho(\alpha) := \int_{-\pi}^{\pi}\left(\operatorname{Re}\log\left(1-\alpha\,\mathrm{e}^{i\,\omega}\right)\right)^2 \mathrm{d}\omega$ with $\alpha := 2\sin\theta$. For $|\theta| \leq \pi/6$ we directly apply Parseval's identity to obtain

$$\rho(2\sin\theta) = \pi\operatorname{Li}_2\left(4\sin^2\theta\right) \tag{78}$$

which is equivalent to (52) since $\rho(\alpha) = 2\pi\rho_2(\alpha)$. In the remaining case we write

$$\begin{aligned}\rho(\alpha) &= \int_{-\pi}^{\pi}\left\{\log|\alpha| + \operatorname{Re}\log\left(1-\alpha^{-1}\,\mathrm{e}^{i\,\omega}\right)\right\}^2 \mathrm{d}\omega\\ &= 2\pi\,\log^2|\alpha| - 2\log|\alpha|\int_{-\pi}^{\pi}\log\left|1-\alpha^{-1}\,\mathrm{e}^{i\,\omega}\right| \mathrm{d}\omega + \pi\operatorname{Li}_2\left(\frac{1}{\alpha^2}\right)\\ &= 2\pi\,\log^2|\alpha| + \pi\operatorname{Li}_2\left(\frac{1}{\alpha^2}\right),\end{aligned} \tag{79}$$

where we have appealed to Parseval's and Jensen's formulae. Thus,

$$\mu_2(1+x+y) = \frac{1}{\pi}\int_0^{\pi/6}\operatorname{Li}_2\left(4\sin^2\theta\right)\mathrm{d}\theta + \frac{1}{\pi}\int_{\pi/6}^{\pi/2}\operatorname{Li}_2\left(\frac{1}{4\sin^2\theta}\right)\mathrm{d}\theta + \frac{\pi^2}{54}, \tag{80}$$

since $\frac{2}{\pi}\int_{\pi/6}^{\pi/2}\log^2\alpha\,\mathrm{d}\theta = \mu(1+x+y_1, 1+x+y_2) = \frac{\pi^2}{54}$. Now, for $\alpha > 1$, the functional equation in [19, A2.1 (6)]

$$\operatorname{Li}_2(\alpha) + \operatorname{Li}_2(1/\alpha) + \frac{1}{2}\,\log^2\alpha = 2\zeta(2) + i\pi\log\alpha \tag{81}$$

gives

$$\int_{\pi/6}^{\pi/2}\left\{\operatorname{Re}\operatorname{Li}_2\left(4\sin^2\theta\right) + \operatorname{Li}_2\left(\frac{1}{4\sin^2\theta}\right)\right\}\mathrm{d}\theta = \frac{5}{54}\pi^3. \tag{82}$$

We then combine (76), (82) and (80) to deduce the desired result (77). □

7.2. Evaluation of $\mu_3(1+x+y)$

In this section we provide a remarkably concise closed form of $\mu_3(1+x+y)$. We were led to this form by the integer relation algorithm PSLQ [1] (see Example 9.2 for some comments on obtaining high precision evaluations), and by considering the evaluation (74) of $\mu_2(1+x+y)$.

The details of formalization are formidable — at least by the route chosen here — and so we proceed more informally leaving three conjectural identities.

Conjecture 7.3 (Evaluation of $\mu_3(1+x+y)$)**.** *We have*

$$\mu_3(1+x+y) \overset{?[1]}{=} \frac{6}{\pi}\operatorname{Ls}_4\left(\frac{2\pi}{3}\right) - \frac{9}{\pi}\operatorname{Cl}_4\left(\frac{\pi}{3}\right) - \frac{\pi}{4}\operatorname{Cl}_2\left(\frac{\pi}{3}\right) - \frac{13}{2}\zeta(3). \tag{83}$$

This evaluation is equivalent to the conjectural identities (43) and (45).

Proof. We first use Theorem 6.3 to write

$$\begin{aligned}\mu_3(1+x+y) &= \frac{2}{\pi}\int_0^{\pi/6}\rho_3(2\sin\theta)\,\mathrm{d}\theta + \frac{2}{\pi}\int_{\pi/6}^{\pi/2}\rho_3\left(\frac{1}{2\sin\theta}\right)\mathrm{d}\theta \\ &\quad + \frac{3}{\pi}\int_{\pi/6}^{\pi/2}\log(2\sin\theta)\operatorname{Li}_2\left(\frac{1}{4\sin^2\theta}\right)\mathrm{d}\theta - \zeta(3) + \frac{9}{2\pi}\operatorname{Cl}_4\left(\frac{\pi}{3}\right),\end{aligned} \tag{84}$$

on appealing to Examples 3.1 and 6.4.

Now the functional equation for the dilogarithm (81) as used above and knowledge of $\operatorname{Ls}_n(\pi/3)$ (see [7, 8]) allow us to deduce

$$\begin{aligned}&\frac{3}{\pi}\int_0^{\pi/6}\log(2\sin\theta)\operatorname{Li}_2\left(4\sin^2\theta\right)\mathrm{d}\theta + \frac{3}{\pi}\int_0^{\pi/6}\log(2\sin\theta)\operatorname{Li}_2\left(\frac{1}{4\sin^2\theta}\right)\mathrm{d}\theta \\ &\qquad = \frac{3}{2}\zeta(3) - \frac{\pi}{2}\operatorname{Cl}_2\left(\frac{\pi}{3}\right) + \frac{27}{2\pi}\operatorname{Cl}_4\left(\frac{\pi}{3}\right),\end{aligned} \tag{85}$$

$$\begin{aligned}&\frac{3}{\pi}\int_{\pi/6}^{\pi/2}\log(2\sin\theta)\operatorname{Re}\operatorname{Li}_2\left(4\sin^2\theta\right)\mathrm{d}\theta + \frac{3}{\pi}\int_{\pi/6}^{\pi/2}\log(2\sin\theta)\operatorname{Re}\operatorname{Li}_2\left(\frac{1}{4\sin^2\theta}\right)\mathrm{d}\theta \\ &\qquad = 3\zeta(3) + \frac{\pi}{2}\operatorname{Cl}_2\left(\frac{\pi}{3}\right) - \frac{27}{2\pi}\operatorname{Cl}_4\left(\frac{\pi}{3}\right).\end{aligned} \tag{86}$$

Moreover, we have

$$\frac{3}{\pi}\left\{\int_0^{\pi/6} + \int_{\pi/6}^{\pi/2}\right\}\log(2\sin\theta)\operatorname{Re}\operatorname{Li}_2\left(4\sin^2\theta\right)\mathrm{d}\theta \overset{?[2]}{=} \frac{7}{2}\zeta(3) - \pi\operatorname{Cl}_2\left(\frac{\pi}{3}\right), \tag{87}$$

$$\frac{3}{\pi}\left\{\int_0^{\pi/6} + \int_{\pi/6}^{\pi/2}\right\}\log(2\sin\theta)\operatorname{Re}\operatorname{Li}_2\left(\frac{1}{4\sin^2\theta}\right)\mathrm{d}\theta \overset{?[2]}{=} \zeta(3) + \pi\operatorname{Cl}_2\left(\frac{\pi}{3}\right), \tag{88}$$

which are provable as was (76) because, for $|z| < 1/2$, we have

$$\frac{1}{\pi}\int_0^{\pi}\log\left(2\sin\frac{\theta}{2}\right)\operatorname{Li}_2\left(4z^2\sin^2\frac{\theta}{2}\right)\mathrm{d}\theta = \sum_{n=1}^{\infty}\binom{2n}{n}\frac{\sum_{k=1}^{2n}\frac{(-1)^k}{k}}{n^2}z^{2n}.$$

(The latter is derivable also from (85), (86) and (87).)

Thence, (85), (86) and (87) together establish that the equality

$$\frac{3}{\pi}\int_{\pi/6}^{\pi/2}\log(2\sin\theta)\,\mathrm{Li}_2\left(\frac{1}{4\sin^2\theta}\right)\mathrm{d}\theta \overset{?[3]}{=} \frac{2}{3}\zeta(3)+\frac{7\pi}{12}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)-\frac{17}{2\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right) \tag{89}$$

is true as soon as we establish

$$I_3 := \frac{3}{\pi}\int_0^{\pi/6}\log(2\sin\theta)\,\mathrm{Li}_2\left(4\sin^2\theta\right)\mathrm{d}\theta \overset{?[3]}{=} \frac{7}{6}\zeta(3)-\frac{11\pi}{12}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+5\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right). \tag{90}$$

This can, in principle, be achieved by writing the integral as

$$I_3 = \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1\frac{s^{2n}}{\sqrt{4-s^2}}\log s\,\mathrm{d}s$$

and using the binomial series to arrive at

$$I_3 = -\frac{3}{2\pi}\sum_{m=0}^{\infty}\frac{\binom{2m}{m}}{4^{2m}}\sum_{n=1}^{\infty}\frac{1}{n^2\left(1+2(n+m)\right)^2}. \tag{91}$$

This leaves us to deal with the two terms in (84) involving ρ_3. These two terms are in turn related by

$$\frac{2}{\pi}\int_0^{\pi/6}\mathrm{Li}_{2,1}\left(4\sin^2\theta\right)\mathrm{d}\theta+\frac{2}{\pi}\int_0^{\pi/6}\mathrm{Re}\,\mathrm{Li}_{2,1}\left(\frac{1}{4\sin^2\theta}\right)\mathrm{d}\theta = \frac{1}{9}\left\{\zeta(3)-\pi\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+\frac{6}{\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\right\}, \tag{92}$$

as we see by integrating (65). Likewise,

$$\frac{2}{\pi}\int_{\pi/6}^{\pi/2}\mathrm{Re}\,\mathrm{Li}_{2,1}\left(4\sin^2\theta\right)\mathrm{d}\theta+\frac{2}{\pi}\int_{\pi/6}^{\pi/2}\mathrm{Li}_{2,1}\left(\frac{1}{4\sin^2\theta}\right)\mathrm{d}\theta = \frac{1}{9}\left\{2\zeta(3)-5\pi\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)-\frac{6}{\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\right\}. \tag{93}$$

Also, using (64) we arrive at

$$\frac{2}{\pi}\int_0^{\pi/6}\mathrm{Li}_{2,1}\left(4\sin^2\theta\right)\mathrm{d}\theta = \frac{20}{27}\zeta(3)-\frac{8\pi}{27}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+\frac{4}{9\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{1}{\pi}\int_0^{\pi/3}\log^2\left(1-4\sin^2\frac{\theta}{2}\right)\log\left(2\sin\frac{\theta}{2}\right)\mathrm{d}\theta, \tag{94}$$

and

$$\frac{2}{\pi}\int_0^{\pi/2}\mathrm{Re}\,\mathrm{Li}_{2,1}\left(4\sin^2\theta\right)\mathrm{d}\theta = \frac{1}{3}\zeta(3)-\frac{2\pi}{3}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right). \tag{95}$$

We may now establish — from (89), (92), (93), (94), (95) and (84) — that

$$\mu_3(1+x+y) = \frac{43}{18}\,\zeta(3) - \frac{47\pi}{36}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{13}{3\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{2}{\pi}\int_0^{\pi/3} \log^2\left(1-4\sin^2\frac{\theta}{2}\right)\log\left(2\sin\frac{\theta}{2}\right)\mathrm{d}\theta. \tag{96}$$

Hence, to prove (83) we are reduced to verifying that

$$-\frac{1}{\pi}\,\mathrm{Ls}_4\left(\frac{2\pi}{3}\right) \overset{?[4]}{=} -\frac{37}{54}\zeta(3) + \frac{7\pi}{27}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{7}{9\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{1}{2\pi}\int_0^{\pi/3} \log^2\left(1-4\sin^2\frac{\theta}{2}\right)\log\left(2\sin\frac{\theta}{2}\right)\mathrm{d}\theta. \tag{97}$$

which completes the evaluation. □

Remark 7.4. By noting that, for integers $n \geq 2$,

$$\mathrm{Cl}_n\left(\frac{\pi}{3}\right) = \left(\frac{1}{2^{n-1}} + (-1)^n\right)\mathrm{Cl}_n\left(\frac{2\pi}{3}\right),$$

the arguments of the Clausen functions in the evaluation (83) of $\mu_3(1+x+y)$ may be transformed to $\frac{2\pi}{3}$.

Many further variations are possible. For instance, it follows from [8] that

$$\mathrm{Ls}_4\left(\frac{2\pi}{3}\right) = \frac{31}{18}\,\pi\zeta(3) + \frac{\pi^2}{12}\,\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) - \frac{3}{2}\,\mathrm{Cl}_4\left(\frac{2\pi}{3}\right) + 6\,\mathrm{Cl}_{2,1,1}\left(\frac{2\pi}{3}\right) \tag{98}$$

in terms of multi Clausen values. ◇

8. Proofs of two conjectures of Boyd

We now use log-sine integrals to recapture the following evaluations conjectured by Boyd in 1998 and first proven in [24] using *Bloch-Wigner* logarithms. Below, L_{-n} denotes a primitive L-series and G is Catalan's constant.

Theorem 8.1 (Two quadratic evaluations)**.** *We have*

$$\mu(y^2(x+1)^2 + y(x^2+6x+1) + (x+1)^2) = \frac{16}{3\pi}\,\mathrm{L}_{-4}(2) = \frac{16}{3\pi}\,\mathrm{G}, \tag{99}$$

as well as

$$\mu(y^2(x+1)^2 + y(x^2-10x+1) + (x+1)^2) = \frac{5\sqrt{3}}{\pi}\,\mathrm{L}_{-3}(2) = \frac{20}{3\,\pi}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right). \tag{100}$$

Proof. Let $P_c = y^2(x+1)^2 + y(x^2+2cx+1) + (x+1)^2$ and $\mu_c = \mu(P_c)$ for a real variable c. We set $x = \mathrm{e}^{2\pi i t}$, $y = \mathrm{e}^{2\pi i u}$ and note that

$$\begin{aligned}|P_c| &= |(x+1)^2(y^2+y+1) + 2(c-1)xy| \\ &= \left|(x+x^{-1}+2)(y+1+y^{-1}) + 2(c-1)\right| \\ &= |2(\cos(2\pi t)+1)(2\cos(2\pi u)+1) + 2(c-1)| \\ &= 2\,|c + 2\cos(2\pi u) + (1+2\cos(2\pi u))\cos(2\pi t)|.\end{aligned}$$

It is known that (see [15, §4.224, Ex. 9]), for real a, b with $|a| \geq |b| > 0$,

$$\int_0^1 \log|2a + 2b\cos(2\pi\theta)|\,\mathrm{d}\theta = \log\left(|a| + \sqrt{a^2-b^2}\right). \tag{101}$$

Applying this, with $a = c + 2\cos(2\pi u)$ and $b = 1 + 2\cos(2\pi u))$ to $\int_0^1 |P_c|\,\mathrm{d}t$, we get

$$\mu_c = \int_0^1 \log\left|c + 2\cos(2\pi u) + \sqrt{(c^2-1) + 4(c-1)\cos(2\pi u)}\right|\,\mathrm{d}u. \tag{102}$$

If $c^2 - 1 = \pm 4(c-1)$, that is if $c = 3$ or $c = -5$, then the surd is a perfect square and also $|a| \geq |b|$.

(a) When $c = 3$ for (99), by symmetry, after factorization we obtain

$$\begin{aligned}\mu_3 &= \frac{1}{\pi}\int_0^\pi \log(1 + 4|\cos\theta| + 4|\cos^2\theta|)\,\mathrm{d}\theta = \frac{4}{\pi}\int_0^{\pi/2} \log(1+2\cos\theta)\,\mathrm{d}\theta \\ &= \frac{4}{\pi}\int_0^{\pi/2} \log\left(\frac{2\sin\frac{3\theta}{2}}{2\sin\frac{\theta}{2}}\right)\mathrm{d}\theta = \frac{4}{3\,\pi}\left(\mathrm{Ls}_2\left(\frac{3\pi}{2}\right) - 3\,\mathrm{Ls}_2\left(\frac{\pi}{2}\right)\right) \\ &= \frac{16}{3}\,\frac{\mathrm{L}_{-4}(2)}{\pi}\end{aligned}$$

as required, since $\mathrm{Ls}_2\left(\frac{3\pi}{2}\right) = -\mathrm{Ls}_2\left(\frac{\pi}{2}\right) = \mathrm{L}_{-4}(2)$, which is Catalan's constant G.

(b) When $c = -5$ for (100), we likewise obtain

$$\begin{aligned}\mu_{-5} &= \frac{2}{\pi}\int_0^\pi \log\left(\sqrt{3} + 2\sin\theta\right)\mathrm{d}\theta = \frac{2}{\pi}\int_{\pi/3}^{4\pi/3} \log\left(\sqrt{3} + 2\sin\left(\theta - \frac{\pi}{3}\right)\right)\mathrm{d}\theta \\ &= \frac{2}{\pi}\int_{\pi/3}^{4\pi/3} \left\{\log 2\left(\sin\frac{\theta}{2}\right) + \log 2\left(\sin\frac{\theta+\frac{\pi}{3}}{2}\right)\right\}\mathrm{d}\theta \\ &= \frac{2}{\pi}\int_{\pi/3}^{4\pi/3} \log 2\left(\sin\frac{\theta}{2}\right)\mathrm{d}\theta + \frac{2}{\pi}\int_{2\pi/3}^{5\pi/3} \log 2\left(\sin\frac{\theta}{2}\right)\mathrm{d}\theta \\ &= \frac{4}{\pi}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{4}{\pi}\,\mathrm{Cl}_2\left(\frac{4\pi}{3}\right) = \frac{20}{3\pi}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right),\end{aligned}$$

since $\mathrm{Cl}_2\left(\frac{4\pi}{3}\right) = -\frac{2}{3}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$ and so we are done. □

When $c = 1$ the cosine in the surd disappears, and we obtain $\mu_1 = 0$, which is trivial as in this case the polynomial factorizes as $(1+x)^2(1+y+y^2)$. For $c = -1$ we are able, with some care, to directly integrate (102) and so to obtain an apparently new Mahler measure:

Theorem 8.2. *We have*

$$\begin{aligned}\mu_{-1} &= \mu\left((x+1)^2(y^2+y+1)-2xy\right) \\ &= \frac{1}{\pi}\left\{\frac{1}{2}B\left(\frac{1}{4},\frac{1}{4}\right){}_3F_2\left(\left.{\frac{1}{4},\frac{1}{4},1 \atop \frac{3}{4},\frac{5}{4}}\right|\frac{1}{4}\right)-\frac{1}{6}B\left(\frac{3}{4},\frac{3}{4}\right){}_3F_2\left(\left.{\frac{3}{4},\frac{3}{4},1 \atop \frac{5}{4},\frac{7}{4}}\right|\frac{1}{4}\right)\right\}.\end{aligned} \tag{103}$$

Here, $B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$ *denotes the Euler beta function.*

We observe that an alternative form of μ_{-1} is given by

$$\mu_{-1} = \mu\left(\left(x+1/x+2\sqrt{1/x}\right)(y+1/y+1)-2\right).$$

Remark 8.3. Equation (101) may be applied to other conjectured Mahler measures. For instance, $\mu(1+x+y+1/x+1/y) = .25133043371325\ldots$ was conjectured by Deninger [14] to evaluate in L-series terms as

$$\mu(1+x+y+1/x+1/y) = 15\sum_{n=1}^{\infty}\frac{a_n}{n^2}, \tag{104}$$

where $\sum_{n=1}^{\infty} a_n q^n = \eta(q)\eta(q^3)\eta(q^5)\eta(q^{15})$. Here η is the *Dirichlet eta-function*:

$$\eta(q) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = q^{1/24}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}. \tag{105}$$

This has recently been proven in [23].

Application of (101) shows that

$$\mu(1+x+y+1/x+1/y) = \frac{1}{\pi}\int_0^{\pi/3}\log\left(\frac{1+2\cos\theta}{2}+\sqrt{\left(\frac{1+2\cos\theta}{2}\right)^2-1}\right)\,d\theta,$$

but the surd remains an obstacle to a direct evaluation. □

9. Conclusion

To recapitulate, $\mu_k(1+x+y) = W_3^{(k)}(0)$ has been evaluated in terms of log-sine integrals for $1 \le k \le 3$. Namely,

$$\mu_1(1+x+y) = \frac{3}{2\pi}\,\mathrm{Ls}_2\left(\frac{2\pi}{3}\right), \tag{106}$$

$$\mu_2(1+x+y) = \frac{3}{\pi}\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{\pi^2}{4}, \tag{107}$$

$$\mu_3(1+x+y) \overset{?[1]}{=} \frac{6}{\pi}\,\mathrm{Ls}_4\left(\frac{2\pi}{3}\right)-\frac{9}{\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)-\frac{\pi}{4}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)-\frac{13}{2}\zeta(3). \tag{108}$$

Hence it is reasonable to ask whether $\mu_4(1+x+y)$ and higher Mahler measures have evaluations in similar terms.

Example 9.1. In the case of $\mu_4(1+x+y)$, numerical experiments suggest that

$$\pi\mu_4(1+x+y) \overset{?[5]}{=} 12\,\mathrm{Ls}_5\left(\frac{2\pi}{3}\right) - \frac{49}{3}\,\mathrm{Ls}_5\left(\frac{\pi}{3}\right) + 81\,\mathrm{Gl}_{4,1}\left(\frac{2\pi}{3}\right) + 3\pi^2\,\mathrm{Gl}_{2,1}\left(\frac{2\pi}{3}\right) + 2\zeta(3)\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \pi\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)^2 - \frac{29}{90}\pi^5 \tag{109}$$

while the higher Mahler measure $\mu_5(1+x+y)$ does not appear to have an evaluation in terms of generalized Glaisher and Clausen values only. ◇

We close with numerical values for these quantities.

Example 9.2. By computing higher-order finite differences in the right-hand side of (23) we have obtained values for $\mu_n(1+x+y)$ to several thousand digits. To confirm these values we have evaluated the double-integral (25) to about 250 digits for all $n \leq 8$. These are the results for $\mu_k := \mu_k(1+x+y)$ to fifty digits:

$$\mu_2 = 0.41929927830117445534618570174886146566170299117521, \tag{110}$$

$$\mu_3 = 0.13072798584098927059592540295887788768895327503289, \tag{111}$$

$$\mu_4 = 0.52153569858138778267996782141801173128244973155094, \tag{112}$$

$$\mu_5 = -0.46811264825699083401802243892432823881642492433794. \tag{113}$$

These values will allow a reader to confirm many of our results numerically. ◇

Acknowledgements We thank David Bailey for his assistance with the two-dimensional quadratures in Example 9.2. Thanks are due to Yasuo Ohno and Yoshitaka Sasaki for introducing us to the relevant papers during their recent visit to CARMA.

References

[1] J. M. Borwein, D. H. Bailey, and R. Girgensohn. *Experimentation in Mathematics: Computational Paths to Discovery.* A. K. Peters, 1st edition, 2004.

[2] J. M. Borwein, D. H. Bailey, and A. Straub. Log-sine evaluations of Mahler measures, III. In preparation, March 2011.

[3] J. M. Borwein and D. M. Bradley. Thirty two Goldbach variations. *Int. J. Number Theory*, 2(1):65–103, 2006.

[4] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisoněk. Special values of multiple polylogarithms. *Trans. Amer. Math. Soc.*, 353(3):907–941, 2001. arXiv:math/9910045.

[5] J. M. Borwein, D. J. Broadhurst, and J. Kamnitzer. Central binomial sums, multiple Clausen values, and zeta values. *Experimental Mathematics*, 10(1):25–34, 2001. arXiv:hep-th/0004153.

[6] J. M. Borwein and R. E. Crandall. Closed forms: what they are and why we care. *Notices Amer. Math. Soc.*, 2010. In press. Available from: http://www.carma.newcastle.edu.au/~jb616/closed-form.pdf.

[7] J. M. Borwein and A. Straub. Log-sine evaluations of Mahler measures. *J. Aust Math. Soc.*, Mar 2011. In Press. arXiv:1103.3893.

[8] J. M. Borwein and A. Straub. Special values of generalized log-sine integrals. *Proceedings of ISSAC 2011 (International Symposium on Symbolic and Algebraic Computation)*, March 2011. arXiv:1103.4298.

[9] J. M. Borwein, A. Straub, J. Wan, and W. Zudilin. Densities of short uniform random walks (with an appendix by Don Zagier). *Canadian Journal of Mathematics*, 2011. In Press. arXiv:1103.2995.

[10] D. W. Boyd. Speculations concerning the range of Mahler's measure. *Canad. Math. Bull.*, 24:453–469, 1981.

[11] A. Davydychev and M. Kalmykov. New results for the ε-expansion of certain one-, two- and three-loop Feynman diagrams. *Nuclear Physics B*, 605:266–318, 2001. arXiv:hep-th/0012189.

[12] A. Davydychev and M. Kalmykov. Massive Feynman diagrams and inverse binomial sums. *Nuclear Physics B*, 699(1-2):3–64, 2004. arXiv:hep-th/0303162.

[13] A. I. Davydychev. Explicit results for all orders of the ε expansion of certain massive and massless diagrams. *Phys. Rev. D*, 61(8):087701, Mar 2000. arXiv:hep-ph/9910224.

[14] S. Finch. Modular forms on $SL_2(\mathbb{Z})$. Preprint, 2005. Available from: http://algo.inria.fr/csolve/frs.pdf.

[15] I. Gradstein and I. Ryszik. *The Tables of Integrals, Sums and Products*. Academic Press, 1980.

[16] M. Kalmykov. About higher order ϵ-expansion of some massive two- and three-loop master-integrals. *Nuclear Physics B*, 718:276–292, July 2005. arXiv:hep-ph/0503070.

[17] M. Kalmykov and A. Sheplyakov. lsjk – a C++ library for arbitrary-precision numeric evaluation of the generalized log-sine functions. *Comput. Phys. Commun.*, 172(1):45–59, 2005. arXiv:hep-ph/0411100.

[18] N. Kurokawa, M. Lalín, and H. Ochiai. Higher Mahler measures and zeta functions. *Acta Arithmetica*, 135(3):269–297, 2008. arXiv:0908.0171.

[19] L. Lewin. On the evaluation of log-sine integrals. *The Mathematical Gazette*, 42:125–128, 1958.

[20] L. Lewin. *Polylogarithms and associated functions*. North Holland, 1981.

[21] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.

[22] K. Onadera. Generalized log sine integrals and the Mordell-Tornheim zeta values. *Trans. Amer. Math. Soc.*, 363(3):1463–1485, 2011.

[23] M. Rogers and W. Zudilin. On the Mahler measure of $1 + x + 1/x + y + 1/y$. Preprint, Feb 2011. arXiv:1102.1153.

[24] N. Touafek. Mahler's measure: proof of two conjectured formulae. *An. St. Univ. Ovidius Constanta*, 16:127–136, 2008.

[25] S. Zlobin. Special values of generalized polylogarithms. To appear in proceedings of the conference "Diophantine and Analytic Problems in Number Theory", 2007. arXiv:0712.1656.