EQUAL KNAPSACK IDENTITIES BETWEEN SYMMETRIC GROUP CHARACTER DEGREES

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Abstract. We prove a series of "knapsack" type equalities for irreducible character degrees of symmetric groups. That is, we find disjoint subsets of the partitions of n so that the two corresponding character-degree sums are equal. Our main result refines our recent description of the Riordan numbers as the sum of all character degrees f^λ where λ is a partition of n into three parts of the same parity. In particular, the sum of the "fat-hook" degrees $f^{(k,k,1^{n-2k})} + f^{(k+1,k+1,1^{n-2k-2})}$ equals the sum of all f^λ where λ has three parts, with the second equal to k and the second and third of equal parity. We further prove an infinite family of additional "knapsack" identities between character degrees.

1. Introduction

Let Σ_n be the symmetric group on n letters and recall that the complex irreducible characters of Σ_n are labelled by partitions of n. The degree of the irreducible character corresponding to a partition $\lambda \vdash n$ is the number of standard Young tableaux of shape λ, which we denote by f^λ.

The famous Catalan numbers $C(n)$ enumerate hundreds of different sets [Sta15], including Dyck paths of semilength n (that is, paths from $(0,0)$ to $(2n,0)$ using steps $U=(1,1)$ and $D=(-1,1)$ and not going below the x-axis). They also count standard Young tableaux of shape (n,n), that is $C(n) = f^{(n,n)}$. Allowing also horizontal steps, one obtains *Motzkin paths*, which are paths from $(0,0)$ to $(n,0)$ using only steps $U=(1,1)$, $F=(1,0)$, and $D=(1,-1)$, and not going below the x-axis. These are enumerated by the Motzkin numbers $M(n)$, sequence A001006 in [OEI25]. $M(n)$ is also well known to count the number of standard Young tableaux of size n with three or fewer rows; for an example of a bijection, see [MV22]. In our notation, this count can be expressed as

$$M(n) = \sum_{\lambda=(\lambda_1,\lambda_2,\lambda_3)\vdash n} f^\lambda. \tag{1.1}$$

Here, and elsewhere, we allow parts to be zero. As such, (1.1) sums over all partitions with up to three nonzero parts.

A *Riordan path* is a Motzkin path with the additional requirement that there may not be a flat step F on the x-axis. We let $R(n)$ be the number of Riordan paths of length n; this is the sequence A005043 in [OEI25]. The Riordan numbers $R(n)$ also have the following interpretation in terms of degrees of irreducible symmetric group characters. This is a comment of Regev given on the OEIS entry (for a proof see [HSW25]).

Proposition 1.1. [HSW25, Proposition 1.2] *Let $0 \le m < n$. The number of Riordan paths of length n with m flat steps and k up steps (and thus k down steps) is $f^{(k,k,1^m)}$.*

In [HSW25], we gave an interpretation of $R(n)$ as the sum of the f^λ, not over all three-part partitions like $M(n)$, but instead over those three-part partitions where the three parts have equal parity (necessarily the parity of n).

Theorem 1.2. [HSW25, Theorem 4.5] *Let $X = \{(\lambda_1,\lambda_2,\lambda_3) \vdash n \mid \lambda_1 \equiv \lambda_2 \equiv \lambda_3 \pmod 2\}$ and let $Y = \{(k,k,1^{n-2k}) \mid 1 \le k \le \lfloor n/2 \rfloor\}$. Then:*

$$\sum_{\lambda\in X} f^\lambda = \sum_{\mu\in Y} f^\mu = R(n). \tag{1.2}$$

We note that the bijection in [MV22] proving (1.1) does not send Riordan paths to tableaux with shapes with parts of equal parity. Indeed, we do not know of a bijective proof of Theorem 1.2.

In this paper we prove that the equality in (1.2) may be refined to a series of equalities, one for each choice of the second part λ_2. For example when $n = 20$ we prove:

$$\begin{aligned} f^{(20)} &= f^{(1^{20})} \\ f^{(18,2)} + f^{(16,2,2)} &= f^{(2,2,1^{16})} + f^{(3,3,1^{14})} \\ f^{(16,4)} + f^{(14,4,2)} + f^{(12,4,4)} &= f^{(4,4,1^{12})} + f^{(5,5,1^{10})} \\ f^{(14,6)} + f^{(12,6,2)} + f^{(10,6,4)} + f^{(8,6,6)} &= f^{(6,6,1^{8})} + f^{(7,7,1^{6})} \\ f^{(12,8)} + f^{(10,8,2)} + f^{(8,8,4)} &= f^{(8,8,1^{4})} + f^{(9,9,1^{2})} \\ f^{(10,10)} &= f^{(10,10)}. \end{aligned}$$

We attempted to prove these identities by induction using the branching rule for symmetric group irreducible characters. However, the branching rule does not preserve the parity conditions, which required us to discover and prove three additional sets of identities all together using a single induction proof. Yet, even these four identities together are not preserved under the branching rules, so attempting a proof by induction leads to error terms which are not subject to the inductive hypothesis. These error terms led to the discovery of two more sets of identities (see Section 2) between character degrees which may be proved directly using the hook length formula, and thus let us prove the four main identities by induction.

These four identities are recorded in the following theorem which is our first main result. Note that the identity (1.3), which always holds when $k \equiv n \pmod 2$, is sufficient for the refinement of Theorem 1.2 which we described above.

Theorem 1.3. *Fix n and k. Let*

$$X_1(n,k) = \{(\lambda_1, k, \lambda_3) \vdash n \mid k \equiv \lambda_3 \pmod 2\}$$

and

$$X_2(n,k) = \{(\lambda_1, k, \lambda_3) \vdash n \mid k \not\equiv \lambda_3 \pmod 2\}.$$

If $k \leq \lceil \frac{n}{3} \rceil$ or if $k \equiv n \pmod 2$ then

$$\sum_{\lambda \in X_1(n,k)} f^\lambda = f^{(k,k,1^{n-2k})} + f^{(k+1,k+1,1^{n-2k-2})} \tag{1.3}$$

as well as

$$\sum_{\lambda \in X_2(n,k)} f^\lambda = f^{(k+1,k,1^{n-2k-1})}. \tag{1.4}$$

If $k > \lceil \frac{n}{3} \rceil$ and $k \not\equiv n \pmod 2$ then the equalities above hold with the roles of X_1 and X_2 switched.

Above, we already illustrated equation (1.3) when $n = 20$ and $k \equiv n \pmod 2$. In the following, we briefly illustrate both of the equalities (1.3) and (1.4) for some additional choices of n and k. For instance, for $n = 32$, $k = 11$, the two equalities become

$$f^{(20,11,1)} + f^{(18,11,3)} + f^{(16,11,5)} + f^{(14,11,7)} + f^{(12,11,9)} = f^{(11,11,1^{10})} + f^{(12,12,1^{8})},$$
$$f^{(21,11,0)} + f^{(19,11,2)} + f^{(17,11,4)} + f^{(15,11,6)} + f^{(13,11,8)} + f^{(11,11,10)} = f^{(12,11,1^{9})}.$$

Similarly, for $n = 32$, $k = 12$, the equalities are

$$f^{(20,12,0)} + f^{(18,12,2)} + f^{(16,12,4)} + f^{(14,12,6)} + f^{(12,12,8)} = f^{(12,12,1^{8})} + f^{(13,13,1^{6})},$$
$$f^{(19,12,1)} + f^{(17,12,3)} + f^{(15,12,5)} + f^{(13,12,7)} = f^{(13,12,1^{7})}.$$

On the other hand, for $n = 32$, $k = 13$, we obtain

$$f^{(18,13,1)} + f^{(16,13,3)} + f^{(14,13,5)} = f^{(14,13,1^5)},$$
$$f^{(19,13,0)} + f^{(17,13,2)} + f^{(15,13,4)} + f^{(13,13,6)} = f^{(13,13,1^6)} + f^{(14,14,1^4)}.$$

In the next section, we prove the two auxiliary identities required in the inductive proof of Theorem 1.3, and then in Section 3 we prove Theorem 1.3. In Section 4, we show that the auxiliary identities of Section 2 can be considerably extended. As an application of these extensions, which are our second main result, we give an alternative proof of identity (1.3) of Theorem 1.3. We conclude by posing additional problems and conjecturing further identities.

2. Auxiliary Identities

In this section we prove two identities between character degrees which appear of independent interest and are a necessary ingredient to proving Theorem 1.3 by induction using the branching rule. Unlike the identities in Theorem 1.3, the two auxiliary identities are of fixed length and so we are able to prove them directly from the hook length formula.

Recall that, for a partition $\lambda \vdash n$ with conjugate partition λ', the number of standard Young tableaux of shape λ is given by the famous *hook length formula* as

$$f^\lambda = \frac{n!}{\prod h_{i,j}},$$

where the product is over the boxes in the Young diagram for λ and $h_{i,j} = \lambda_i + \lambda'_j - i - j + 1$ is the hook length at position (i, j).

For diagrams of shape $(a, b, 1^t)$ or (r, s, t) it is easy to write down all the hook lengths. For example the hook lengths in the first row of shape (r, s, t) are:

$$\{r+2, r+1, r, \ldots, r-t+3, r-t+1, r-t, r-t-1, \ldots, r-s+2, r-s, r-s-1, \ldots, 1\}$$

so their product is $\frac{(r+2)!}{(r-t+2)(r-s+1)}$. Using the hook length formula one easily obtains the following explicit formulas.

Lemma 2.1. *For $a \geq b \geq 1$, $t \geq 0$,*

$$f^{(a,b,1^t)} \ = \ \frac{(a+b+t)!\,(a-b+1)}{(a+t+1)(b+t)\;a!\,(b-1)!\;t!}.$$

Lemma 2.2. *For $r \geq s \geq t \geq 0$,*

$$f^{(r,s,t)} \ = \ \frac{(r+s+t)!(r-s+1)(r-t+2)(s-t+1)}{(r+2)!(s+1)!\,t!}.$$

The identities in Theorem 1.3 are relationships between various f^λ for partitions λ of these shapes. The following auxiliary identity is a fixed-length relationship of that kind as well. Fix integers $k \geq 2$ and $m \geq 4$ and define

$$L(k, m) = f^{(k,k,1^m)} + f^{(k+1,k+1,1^{m-2})} + f^{(k+2,k+2,1^{m-4})}. \tag{2.1}$$

Then we have:

Lemma 2.3. *For $k \geq 2$, $m \geq 4$,*

$$L(k, m) = f^{(k+2,k,1^{m-2})} + \begin{cases} f^{(k,k,m)}, & m \leq k, \\ 0, & m = k+1 \ \textit{or}\ m = k+2, \\ f^{(m-2,k+1,k+1)}, & m \geq k+3. \end{cases}$$

The following three examples, all for $n = 35$, illustrate each of the three cases from Lemma 2.3:

$$f^{(14,14,1^7)} + f^{(15,15,1^5)} + f^{(16,16,1^3)} = f^{(16,14,1^5)} + f^{(14,14,7)},$$
$$f^{(11,11,1^{13})} + f^{(12,12,1^{11})} + f^{(13,13,1^9)} = f^{(13,11,1^{11})},$$
$$f^{(7,7,1^{21})} + f^{(8,8,1^{19})} + f^{(9,9,1^{17})} = f^{(9,7,1^{19})} + f^{(19,8,8)}.$$

Proof. Consider the difference

$$\Delta_3(k,m) = f^{(k,k,1^m)} + f^{(k+1,k+1,1^{m-2})} + f^{(k+2,k+2,1^{m-4})} - f^{(k+2,k,1^{m-2})}$$

and apply Lemma 2.1 to each term. We obtain, in particular,

$$\begin{aligned} f^{(k,k,1^m)} &= \frac{(2k+m)!}{k!(k-1)!m!} \frac{1}{(k+m)(k+m+1)}, \\ f^{(k+2,k,1^{m-2})} &= \frac{(2k+m)!}{(k+2)!(k-1)!(m-2)!} \frac{3}{(k+m+1)(k+m-2)}. \end{aligned} \tag{2.2}$$

For each of the four terms f^λ of $\Delta_3(k,m)$, we find that $f^\lambda / f^{(k,k,1^m)}$ is a rational function in k and m. For instance,

$$\frac{f^{(k+2,k,1^{m-2})}}{f^{(k,k,1^m)}} = \frac{3(k+m)(m-1)m}{(k+1)(k+2)(k+m-2)}.$$

Consequently, $\Delta_3(k,m)/f^{(k,k,1^m)}$ is a rational function in k and m as well. Indeed, this rational function factors as

$$\frac{\Delta_3(k,m)}{f^{(k,k,1^m)}} = \frac{(k-m+1)(k-m+2)(k+m)(k+m+1)}{k(k+1)^2(k+2)}.$$

Multiplying both sides with $f^{(k,k,1^m)}$ as evaluated in (2.2), we obtain

$$\Delta_3(k,m) = \frac{(2k+m)!}{(k+1)!(k+2)!m!}(k-m+1)(k-m+2). \tag{2.3}$$

This shows that $\Delta_3(k,m) = 0$ if $k = m-1$ or $k = m-2$, as claimed. On the other hand, by the hook length formula as in Lemma 2.2 we have

$$f^{(m-2,k+1,k+1)} = \frac{(2k+m)!(m-k-2)(m-k-1)}{m!(k+2)!(k+1)!}$$

provided that $m-2 \le k+1$. Since the right-hand side equals (2.3) this proves the case $m \ge k+3$. Likewise applying Lemma 2.2, we find that $f^{(k,k,m)}$ evaluates to (2.3) as well provided that $m \le k$. □

Theorem 1.3 splits into cases depending on how the size of k compares to $n/3$ and the respective parities of k and n. When k is almost equal to $n/3$ we get a sort of boundary case in the induction, which needs the following identity.

Proposition 2.4. *Let $k = m \pm 1$. Then:*

$$f^{(k,k,1^m)} + f^{(k+1,k+1,1^{m-2})} = f^{(k+1,k,1^{m-1})}.$$

Proof. One readily verifies from Lemma 2.1 that

$$f^{(k,k,1^m)} + f^{(k+1,k+1,1^{m-2})} - f^{(k+1,k,1^{m-1})} = \frac{(2k+m)!\,(k-m-1)(k-m+1)}{(k+m-1)(k+m+1)\,(k+1)!\,k!\,m!}. \tag{2.4}$$

The claimed identity then follows because the right-hand side of (2.4) is zero when $k = m \pm 1$. □

3. Proof of Theorem 1.3 via the branching rule

Recall that the branching rule describes how irreducible characters of the symmetric group Σ_n decompose upon restriction to Σ_{n-1}. For $\lambda \vdash n$, the dimension f^λ is equal to the sum of all the f^μ where $\mu \vdash n-1$ is obtained by removing a box from the Young diagram of λ in a way that leaves a valid partition of $n-1$.

The proof of Theorem 1.3 is by induction on n. Since small values of n are easily checked, we assume that all possible equalities from the theorem hold for all values smaller than n. We write down each of the desired equalities and apply the branching rule on both sides. On the right-hand side we will get four terms from (1.3) or three terms from (1.4). On the left-hand side we look at the partitions in $X_1(n,k)$ or $X_2(n,k)$, apply the branching theorem, and divide the result into three summands based on whether boxes were removed from the first, second or third rows.

About half the time all three rows give sets of the form $X_i(n-1,k)$ (rows 1 and 3) or $X_i(n-1,k-1)$ (row 2) for $i = 0$ or 1, so the inductive hypotheses apply to all three terms and give the result immediately. The other half of the time there is a term of the form $f^{(a,a,b)}$ or $f^{(a,b,b)}$ that is missing from one of the resulting sums, and we need to apply Lemma 2.3. In one boundary case when $3 \mid n$ and $k = \frac{n}{3} + 1$ we also need Proposition 2.4.

Let's begin by looking at some illustrative examples. Suppose $n = 20$ and $k = 5$. To prove (1.3) we need to show that

$$f^{(14,5,1)} + f^{(12,5,3)} + f^{(10,5,5)} = f^{(5,5,1^{10})} + f^{(6,6,1^8)}. \tag{3.1}$$

Applying the branching rule on the right-hand side we get

$$f^{(5,4,1^{10})} + f^{(5,5,1^9)} + f^{(6,5,1^8)} + f^{(6,6,1^7)}.$$

To likewise apply the branching rule on the left-hand side of (3.1), we remove boxes from one row at a time from the involved partitions. Removing from the first row, we obtain $f^{(13,5,1)} + f^{(11,5,3)} + f^{(9,5,5)}$ which equals $f^{(5,5,1^9)} + f^{(6,6,1^7)}$ by the induction hypothesis. Removing instead from the second row, we get $f^{(14,4,1)} + f^{(12,4,3)}$ which equals $f^{(5,4,1^{10})}$ by induction. Finally, removing from the third row yields $f^{(14,5)} + f^{(12,5,2)} + f^{(10,5,4)}$ which is $f^{(6,5,1^8)}$ by induction. The total from the left-hand side also is $f^{(5,4,1^{10})} + f^{(5,5,1^9)} + f^{(6,5,1^8)} + f^{(6,6,1^7)}$, thus proving the identity (3.1).

To likewise prove (1.4) for the same n and k, we need to show that

$$f^{(15,5)} + f^{(13,5,2)} + f^{(11,5,4)} = f^{(6,5,1^9)}. \tag{3.2}$$

Applying the branching rule on the right-hand side we get

$$f^{(5,5,1^9)} + f^{(6,4,1^9)} + f^{(6,5,1^8)}.$$

To apply the branching rule on the left-hand side, we again remove boxes from one row at a time. Removing from the first row results in $f^{(14,5)} + f^{(12,5,2)} + f^{(10,5,4)}$ which equals $f^{(6,5,1^8)}$ by the induction hypothesis. Removing instead from the second row, we obtain $f^{(15,4)} + f^{(13,4,2)} + f^{(11,4,4)}$ which equals $f^{(4,4,1^{11})} + f^{(5,5,1^9)}$ by induction. Removing from the third row, we get $f^{(13,5,1)} + f^{(11,5,3)}$. The inductive hypothesis for $X_1(19,5)$ gives a formula for $f^{(13,5,1)} + f^{(11,5,3)} + f^{(9,5,5)}$. The partition $(9,5,5)$ lies in $X_1(19,5)$ but does not show up in the branching (it would have to come from $(9,5,6)$ which is not a partition.) By the induction hypothesis, the terms resulting from removing from the third row therefore are

$$f^{(5,5,1^9)} + f^{(6,6,1^7)} - f^{(9,5,5)}.$$

We are left to confirm that

$$f^{(5,5,1^9)} + f^{(6,4,1^9)} + f^{(6,5,1^8)} = f^{(6,5,1^8)} + f^{(4,4,1^{11})} + f^{(5,5,1^9)} + f^{(5,5,1^9)} + f^{(6,6,1^7)} - f^{(9,5,5)}.$$

Cancelling the terms that appear on both sides and applying Lemma 2.3 for $k = 4, m = 11$, we readily confirm this equality, thus completing the proof of (1.4) in the present example.

Suppose we want to instead prove (1.4) for $n = 14$ and $k = 5$ (so that $k = \lceil n/3 \rceil$). This is a "boundary case" and find that the expected "missing term" does not appear. In this case, we need to show that

$$f^{(9,5)} + f^{(7,5,2)} + f^{(5,5,4)} = f^{(6,5,1^3)}.$$

When we remove from the third rows on the left-hand side, we get $f^{(7,5,1)} + f^{(5,5,3)}$. This matches $X_1(13, 5) = \{(7, 5, 1), (5, 5, 3)\}$ so that there is not a missing term of the form $f^{(a,5,5)}$ (for contrast, note that for $n = 16$ the set $X_1(15, 5)$ contains $(5, 5, 5)$ and this partition is missing in the branching). Continuing to branch as in the previous examples, we end up needing to prove that

$$f^{(5,5,1^3)} + f^{(6,4,1^3)} + f^{(6,5,1^2)} = f^{(6,5,1^2)} + f^{(4,4,1^5)} + f^{(5,5,1^3)} + f^{(5,5,1^3)} + f^{(6,6,1)}.$$

This reduces to

$$f^{(6,4,1^3)} = f^{(4,4,1^5)} + f^{(5,5,1^3)} + f^{(6,6,1)},$$

which is exactly the case $k = 4, m = 5$ where the second term in Lemma 2.3 vanishes. This happens precisely when that term is not needed to cancel anything out. This illustrates how the three cases in Lemma 2.3 are precisely calibrated to the needs of our inductive proof.

Finally, there is a boundary case where $n = 3x$ and $k = x + 1$ (so that $k > \lceil \frac{n}{3} \rceil = x$). This is the "large k" regime of Theorem 1.3 where $k \not\equiv n \pmod 2$. On the other hand, when we branch by removing from the second row, the value of k drops to x and $\lceil \frac{n-1}{3} \rceil = x$ which is a switch to the "small k" regime. In this case, we need Proposition 2.4. For example, suppose $n = 24$ and $k = 9$ and we try to prove (1.3), which is

$$f^{(14,9,1)} + f^{(12,9,3)} + f^{(10,9,5)} = f^{(10,9,1^5)}.$$

Applying the branching rule to both sides as before and cancelling equal terms, we end up having to prove that

$$f^{(10,10,1^3)} + f^{(9,8,1^6)} = f^{(10,8,1^5)} + f^{(8,8,7)}.$$

By Lemma 2.3 the right-hand side is just $f^{(8,8,1^7)} + f^{(9,9,1^5)} + f^{(10,10,1^3)}$ so we end up needing to show that

$$f^{(9,8,1^6)} = f^{(8,8,1^7)} + f^{(9,9,1^5)},$$

which is an instance of Proposition 2.4.

At this point, we could safely leave the general details to the reader. Theorem 1.3 contains two equalities which are flipped when $k > \lceil \frac{n}{3} \rceil$ and $k \not\equiv n \pmod 2$. So one has to consider the cases where k is quite large or quite small compared to $n/3$, where it is the same or opposite parity as n, and for both equalities. These are eight cases already. Then, as we indicated above, there are boundary cases when k is close to $n/3$ where the middle case of Lemma 2.3 or Proposition 2.4 are used.

We provide details for three of these cases of Theorem 1.3 in Appendix A and leave the remaining cases for the reader. On the other hand, we provide in the next section an alternative analytical proof of the part of Theorem 1.3 that refines Theorem 1.2.

4. An extension and an alternative proof

Let us consider the sums

$$L_d(k, m) = \sum_{j=0}^{d-1} f^{(k+j,k+j,1^{m-2j})}.$$

In terms of this notation, Lemma 2.3 shows that

$$L_3(k, m) = f^{(k+2,k,1^{m-2})} + \begin{cases} f^{(k,k,m)}, & \text{if } m \le k, \\ 0, & \text{if } m = k+1 \text{ or } m = k+2, \\ f^{(m-2,k+1,k+1)}, & \text{if } m \ge k+3. \end{cases}$$

In this section, we show that this can be extended to the sums $L_d(k,m)$ for all odd d. We then demonstrate that this extension provides an alternative approach to the refinement of Theorem 1.2 proved in Theorem 1.3.

Before turning to the general case, we highlight the case $d=5$.

Example 4.1. Let $k \geq 2$ and $m \geq 4$. If $m \leq k$ then

$$L_5(k,m) = f^{(k+4,k,1^{m-4})} + f^{(k,k,m)} + f^{(k+2,k,m-2)} + f^{(k+2,k+2,m-4)}. \tag{4.1}$$

On the other hand, if $m \geq k+9$ then

$$L_5(k,m) = f^{(k+4,k,1^{m-4})} + f^{(m-2,k+1,k+1)} + f^{(m-4,k+3,k+1)} + f^{(m-6,k+3,k+3)}. \tag{4.2}$$

For each of the intermediate cases $m = k+\delta$ with $\delta \in \{1,2,\ldots,8\}$ a suitable variation of the identities (4.1) and (4.2) holds. Specifically:

$$\begin{aligned}
L_5(k,k+1) &= f^{(k+4,k,1^{k-3})} + f^{(k+2,k,k-1)} + f^{(k+2,k+2,k-3)}\\
L_5(k,k+2) &= f^{(k+4,k,1^{k-2})} + f^{(k+2,k,k)} + f^{(k+2,k+2,k-2)}\\
L_5(k,k+3) &= f^{(k+4,k,1^{k-1})} + f^{(k+1,k+1,k+1)} + f^{(k+2,k+2,k-1)}\\
L_5(k,k+4) &= f^{(k+4,k,1^{k})} + f^{(k+2,k+2,k)}\\
L_5(k,k+5) &= f^{(k+4,k,1^{k+1})} + f^{(k+3,k+1,k+1)}\\
L_5(k,k+6) &= f^{(k+4,k,1^{k+2})} + f^{(k+4,k+1,k+1)} + f^{(k+2,k+2,k+2)}\\
L_5(k,k+7) &= f^{(k+4,k,1^{k+3})} + f^{(k+5,k+1,k+1)} + f^{(k+3,k+3,k+1)}\\
L_5(k,k+8) &= f^{(k+4,k,1^{k+4})} + f^{(k+6,k+1,k+1)} + f^{(k+4,k+3,k+1)}
\end{aligned}$$

Note that the cases $\delta = 3,6$ are combining terms from both of the identities (4.1) and (4.2) in the theorem (whereas the other cases result from one of the identities by removing those terms f^λ for which λ is not a partition).

The following generalizes the case $m \leq k$ of Lemma 2.3. The corresponding case of large m is obtained in Corollary 4.5.

Theorem 4.2. *Let $d \geq 0$ be an integer. Let $k,m \geq 2$ be integers such that $m \geq 4(d-1)$. Then, for $m \leq k$,*

$$L_{2d+1}(k,m) = f^{(k+2d,k,1^{m-2d})} + \sum_{r=0}^{d-1}\sum_{j=0}^{r} f^{(k+2r,k+2j,m-2(r+j))}. \tag{4.3}$$

Proof. The conditions on k and m are only needed to guarantee that each term in (4.3) is of the form f^λ for some partition λ. In the argument below, we actually need to assume the slightly stronger condition $m \geq 4(d+1)$ in order that all of the appearing terms f^λ (such as in equation (4.5)) are such that λ is a partition. We refer to Remark 4.4 for how to remove this additional restriction.

The case $d=0$ of (4.3) is true because both sides equal $f^{(k,k,1^m)}$ by definition. If we subtract the identity for $L_{2d+1}(k,m)$ from the corresponding identity for $L_{2d+3}(k,m)$, we obtain

$$\begin{aligned}
&f^{(k+2d+1,k+2d+1,1^{m-4d-2})} + f^{(k+2d+2,k+2d+2,1^{m-4d-4})}\\
&= f^{(k+2d+2,k,1^{m-2d-2})} - f^{(k+2d,k,1^{m-2d})} + \sum_{j=0}^{d} f^{(k+2d,k+2j,m-2(d+j))}.
\end{aligned} \tag{4.4}$$

By induction, the identity (4.3) is true for all $d \geq 0$ if we can show that (4.4) holds for all $d \geq 0$.

In the remainder, we prove (4.4) by induction on d. The base case $d=0$ of (4.4) reads

$$f^{(k+1,k+1,1^{m-2})} + f^{(k+2,k+2,1^{m-4})} = f^{(k+2,k,1^{m-2})} - f^{(k,k,1^m)} + f^{(k,k,m)}$$

which is equivalent to Lemma 2.3. Observe that replacing d by $d-1$ as well as k by $k+2$ and m by $m-4$ in (4.4) results in:

$$\begin{aligned} & f^{(k+2d+1,k+2d+1,1^{m-4d-2})} + f^{(k+2d+2,k+2d+2,1^{m-4d-4})} \\ = \; & f^{(k+2d+2,k+2,1^{m-2d-4})} - f^{(k+2d,k+2,1^{m-2d-2})} + \sum_{j=0}^{d-1} f^{(k+2d,k+2j+2,m-2(d+j+1))} \end{aligned} \tag{4.5}$$

Let us fix a value $d > 0$ with the aim of proving (4.4). Since (4.5) was obtained from (4.4) by replacing d with $d-1$, we may assume as the induction hypothesis that (4.5) holds for our fixed value of d. Note that the left-hand sides of (4.4) and (4.5) agree. Further note that the terms of the sum over j in (4.5) also appear in the corresponding sum in (4.4), except that the index j is shifted and that (4.4) has an additional term. Accordingly, subtracting (4.5) from (4.4) results in

$$\begin{aligned} 0 \; = \; & f^{(k+2d+2,k,1^{m-2d-2})} - f^{(k+2d,k,1^{m-2d})} + f^{(k+2d,k,m-2d)} \\ & - f^{(k+2d+2,k+2,1^{m-2d-4})} + f^{(k+2d,k+2,1^{m-2d-2})}, \end{aligned}$$

and this fixed-length identity is equivalent to (4.4). If we replace m by $m+2d$ and then replace $2d$ by $\ell - k$ the identity takes the form

$$0 \; = \; f^{(\ell+2,k,1^{m-2})} - f^{(\ell,k,1^{m})} + f^{(\ell,k,m)} - f^{(\ell+2,k+2,1^{m-4})} + f^{(\ell,k+2,1^{m-2})}.$$

This identity is proven in Lemma 4.3 below. □

The following identity completes the proof of Theorem 4.2. We record it separately here since it appears to have independent value (for instance, in light of Remark 4.4, the special case $\ell = k$ can be seen to be equivalent to Lemma 2.3).

Lemma 4.3. *For integers $k \geq m \geq 4$ and $\ell \geq k+2$,*

$$f^{(\ell,k,m)} = f^{(\ell,k,1^m)} - f^{(\ell,k+2,1^{m-2})} - f^{(\ell+2,k,1^{m-2})} + f^{(\ell+2,k+2,1^{m-4})}. \tag{4.6}$$

Proof. We proceed similarly to the proof of Lemma 2.3 and apply the hook length formula as in Lemma 2.1 to each term on the right-hand side of (4.6). That is, we write

$$f^{(\ell,k,1^m)} \; = \; \frac{(\ell+k+m)!}{\ell!(k-1)!m!} \frac{(\ell-k+1)}{(k+m)(\ell+m+1)}, \tag{4.7}$$

with corresponding shifted expressions for the other three terms on the right-hand side of (4.6). Similarly applying Lemma 2.2, we find

$$f^{(\ell,k,m)} = \frac{(\ell+k+m)!}{(\ell+2)!(k+1)!m!} (\ell-k+1)(\ell-m+2)(k-m+1).$$

For each of the four terms f^λ on the right-hand side of (4.6), we find that $f^\lambda / f^{(\ell,k,m)}$ is a rational function in ℓ, k and m. For instance, for the first term

$$\frac{f^{(\ell,k,1^m)}}{f^{(\ell,k,m)}} = \frac{(\ell+1)(\ell+2)k(k+1)}{(k+m)(\ell+m+1)(\ell-m+2)(k-m+1)}.$$

We then observe that the combination of the four rational functions as in (4.6) simplifies to 1. More precisely, we find

$$\frac{f^{(\ell,k,1^m)}}{f^{(\ell,k,m)}} - \frac{f^{(\ell,k+2,1^{m-2})}}{f^{(\ell,k,m)}} - \frac{f^{(\ell+2,k,1^{m-2})}}{f^{(\ell,k,m)}} + \frac{f^{(\ell+2,k+2,1^{m-4})}}{f^{(\ell,k,m)}} = 1,$$

which proves (4.6). □

Remark 4.4. Define the functions

$$\begin{aligned} h(x,y,z) &= \frac{(x+y+z)!}{(x+2)!(y+1)!z!}\,(x-y+1)(x-z+2)(y-z+1), \\ h_1(x,y;z) &= \frac{(x+y+z)!}{x!(y-1)!z!}\,\frac{(x-y+1)}{(x+z+1)(y+z)}. \end{aligned}$$

Apart from singularities, these are defined for any complex value of x, y, z (here, as usual, the factorial $x!$ is defined as $\Gamma(x+1)$ where Γ is the gamma function). On the other hand, note that, up to labelling of the variables, these match the expressions obtained in Lemmas 2.1 and 2.2 using the hook length formula. In other words, if $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a partition then we have

$$f^{(\lambda_1,\lambda_2,\lambda_3)} = h(\lambda_1, \lambda_2, \lambda_3)$$

and similarly

$$f^{(\lambda_1,\lambda_2,1^r)} = h_1(\lambda_1, \lambda_2; r).$$

Instead of working with f^λ for partitions λ, we can replace f^λ with the appropriate quantities $h(\lambda_1, \lambda_2, \lambda_3)$ and $h_1(\lambda_1, \lambda_2; r)$ in the proof of Theorem 4.2 as well as Lemma 4.3. Doing so, we find that the proof of Theorem 4.2 extends to establishing the identity

$$\begin{aligned} &\sum_{j=0}^{2d} h_1(k+j, k+j; m-2j) \\ =\ & h_1(k+2d, k; m-2d) + \sum_{r=0}^{d-1}\sum_{j=0}^{r} h(k+2r, k+2j, m-2(r+j)) \end{aligned} \tag{4.8}$$

where $d \geq 0$ is an integer and where k and m are complex variables. The conditions on k and m in Theorem 4.2 are such that each of the terms involving h or h_1 can be interpreted as f^λ for some partition λ.

Theorem 4.2 provides an evaluation of $L_{2d+1}(k, m)$ when $m \leq k$. The analytic perspective of Remark 4.4 allows us to obtain a corresponding evaluation when m is large.

Corollary 4.5. *Let $d \geq 0$ as well as $k, m \geq 2$ be integers. If $m \geq k + 6d - 3$ then*

$$L_{2d+1}(k,m) = f^{(k+2d,k,1^{m-2d})} + \sum_{r=0}^{d-1}\sum_{j=0}^{r} f^{(m-2(r+j+1),k+2r+1,k+2j+1)}. \tag{4.9}$$

Proof. From the definition of $h(x, y, z)$ in Remark 4.4 it is readily observed that

$$h(x,y,z) = h(z-2, x+1, y+1). \tag{4.10}$$

In particular,

$$h(k+2r, k+2j, m-2(r+j)) = h(m-2(r+j)-2, k+2r+1, k+2j+1).$$

Applying this relation to the identity (4.8) results in identity (4.9) provided that each term involving h is of the form $h(\lambda)$ where λ is a partition (and likewise for h_1). This is ensured by the condition $m \geq k + 6d - 3$ which implies that the part $m - 2(r + j + 1)$ is no less than the part $k + 2r + 1$ for all relevant choices of r and j. □

In the case $d = 1$, Corollary 4.5 reduces to the second case of Lemma 2.3 while the case $d = 2$ results in the identity (4.2). The intermediate identities discussed in Example 4.1 can be obtained by inspecting each term of the corresponding analytic identity (4.8).

As an application, we use the results of this section to give an alternative proof of the part of Theorem 1.3 that refines Theorem 1.2.

Theorem 4.6. *If $k \le \lceil n/3 \rceil$ or if $n \equiv k \pmod 2$, then*

$$\sum_{\lambda \in X_1(n,k)} f^\lambda = f^{(k,k,1^{n-2k})} + f^{(k+1,k+1,1^{n-2k-2})} \tag{4.11}$$

where $X_1(n,k) = \{(\lambda_1, k, \lambda_3) \vdash n \,|\, k \equiv \lambda_3 \pmod 2\}$ as in Theorem 1.3.

Proof. Set $m = n - 2k$. Start with $L_{2d+3}(k,m) - L_{2d+1}(k,m)$ and apply identity (4.9) to obtain

$$\begin{aligned} & f^{(k+2d+1,k+2d+1,1^{m-4d-2})} + f^{(k+2d+2,k+2d+2,1^{m-4d-4})} \\ = \quad & f^{(k+2d+2,k,1^{m-2d-2})} - f^{(k+2d,k,1^{m-2d})} + \sum_{j=0}^{d} f^{(m-2(d+j+1),k+2d+1,k+2j+1)}. \end{aligned}$$

As explained in Remark 4.4 this identity holds without restriction on k and m if we replace f by the complex functions h and h_1. If we further replace k by $k-2d-1$ and m by $m+4d+2$, we obtain

$$\begin{aligned} & f^{(k,k,1^m)} + f^{(k+1,k+1,1^{m-2})} \qquad (4.12) \\ = \quad & h_1(k+1, k-2d-1; m+2d) - h_1(k-1, k-2d-1; m+2d+2) + \sum_{j=0}^{d} h(m+2j, k, k-2j) \end{aligned}$$

where we replaced j by $d-j$ in the final sum. By definition,

$$h_1(k+1, k-2d-1; m+2d) = \frac{(m+2k)!}{(k+1)!(k-2d-2)!(m+2d)!} \frac{2d+3}{(m+k+2d+2)(m+k-1)}$$

and, because of the factor $(k-2d-2)!$ in the denominator, this expression vanishes for both $d = k/2$ and $d = (k-1)/2$. Likewise, the term $h_1(k-1, k-2d-1; m+2d+2)$ vanishes for both $d = k/2$ and $d = (k-1)/2$. Hence, specializing (4.12) to $d = \lfloor k/2 \rfloor$ results in

$$f^{(k,k,1^m)} + f^{(k+1,k+1,1^{m-2})} = \sum_{j=0}^{\lfloor k/2 \rfloor} h(m+2j, k, k-2j). \tag{4.13}$$

If $m \ge k$ or, equivalently, $3k \le n$ then the summands on the right-hand side are $f^{(m+2j,k,k-2j)}$ and the sum equals the sum over f^λ for $\lambda \in X_1(n,k)$. This proves (4.11) in the case $k \le n/3$. To show that (4.11) continues to hold for $k \le \lceil n/3 \rceil$ or, equivalently, $3k \le n+2$, we need to also consider the cases $m = k-1$ and $m = k-2$. In those cases, the right-hand side of (4.13) equals $h(m,k,k)$ plus the sum over f^λ for $\lambda \in X_1(n,k)$. In other words, if $m = k-1$ or $m = k-2$ then

$$f^{(k,k,1^m)} + f^{(k+1,k+1,1^{m-2})} = h(m,k,k) + \sum_{\lambda \in X_1(n,k)} f^\lambda.$$

By the definition of h, we see that

$$h(m,k,k) = \frac{(m+2k)!}{(m+2)!(k+1)!k!} (m-k+1)(m-k+2)$$

vanishes for $m = k-1$ as well as $m = k-2$. This completes the proof of (4.11) in the case $k \le \lceil n/3 \rceil$.

In the remainder, we therefore assume that $3k > n$ and $n \equiv k \pmod 2$. Equivalently, in terms of m, we assume $k > m$ and $m \equiv k \pmod 2$. Note that the summand $s(j) = h(m+2j, k, k-2j)$ of (4.13) satisfies the symmetry

$$s\left(\frac{k-m}{2} - 1 - j\right) = -s(j)$$

which follows immediately from the representation

$$s(j) = \frac{(m+2k)!(m+2j-k+1)(m-k+4j+2)(2j+1)}{(m+2j+2)!(k+1)!(k-2j)!}.$$

Consequently,

$$\sum_{j=0}^{(k-m)/2-1} s(j) = \sum_{j=0}^{(k-m)/2-1} s\left(\frac{k-m}{2}-1-j\right) = -\sum_{j=0}^{(k-m)/2-1} s(j)$$

so that these sums all vanish. Combined with (4.13), we therefore have

$$f^{(k,k,1^m)} + f^{(k+1,k+1,1^{m-2})} = \sum_{j=(k-m)/2}^{\lfloor k/2 \rfloor} h(m+2j,k,k-2j) = \sum_{\lambda\in X_1(n,k)} f^\lambda$$

as claimed. □

The following examples illustrate the two different cases where (4.13) has additional terms compared to (4.11).

Example 4.7. Let $n = 13$ and $k = 5$ so that $k > n/3$ but $k \le \lceil n/3 \rceil$. Then (4.13) takes the form

$$f^{(5,5,1^3)} + f^{(6,6,1^1)} = h(3,5,5) + h(5,5,3) + h(7,5,1).$$

Since $h(3,5,5) = 0$, this reduces to $f^{(5,5,1^3)} + f^{(6,6,1^1)} = f^{(5,5,3)} + f^{(7,5,1)}$ which is (4.11).

Example 4.8. To illustrate the case $k > n/3$ and $n \equiv k \pmod 2$, let $n = 20$ and $k = 8$. In that case the identity (4.13) becomes

$$\begin{aligned} f^{(8,8,1^4)} + f^{(9,9,1^2)} &= \sum_{j=0}^{4} h(4+2j,8,8-2j) \\ &= h(4,8,8) + h(6,8,6) + f^{(8,8,4)} + f^{(10,8,2)} + f^{(12,8)}. \end{aligned}$$

The terms $h(4,8,8) = 1385670$ and $h(6,8,6) = -1385670$ cancel and we arrive at

$$f^{(8,8,1^4)} + f^{(9,9,1^2)} = f^{(8,8,4)} + f^{(10,8,2)} + f^{(12,8)}$$

as predicted by (4.11).

It is natural to wonder if the results in this section extend to the sums $L_d(k,m)$ when d is even. For general values of k and m, our numerical data suggests that no analogous identities exist. However, more specialized formulas appear to hold when k and m have opposite parity. We do not pursue this question here.

5. Concluding remarks

Our main results, Theorem 1.3 and Theorem 4.2, as well as auxiliary results such as Lemma 2.3, are all instances of identities between character degrees of the form

$$\sum_{\lambda\in X_n} f^\lambda = \sum_{\mu\in Y_n} f^\mu \tag{5.1}$$

where X_n and Y_n are certain sets of partitions of size n. One can think of (5.1) as packing two “knapsacks” with the various λ as books and the f^λ as the corresponding weights so that both knapsacks are filled to equal weight. It is natural to wonder if one can find further interesting results of the form (5.1) for natural sets X_n and Y_n of partitions.

There is only one place we have seen results of the form (5.1), namely in the context of modular representation theory. Starting with a partition μ and an integer k, if we add a single rim k-hook to μ in every possible way, we get an alternating sum of characters that vanishes on any permutation not containing a k-cycle. In particular, it vanishes on the identity permutation, so that we obtain an identity on the character degrees. For details, we refer to [Jam84, 21.7]. For an example, if $\mu = (3,1)$ and $k = 6$ then we obtain

$$f^{(9,1)} - f^{(6,4)} + f^{(4,4,2)} - f^{(3,3,2,1^2)} + f^{(3,2,2,1^3)} - f^{(3,1^7)} = 0.$$

Can our identities be related to those arising from this hook-wrapping procedure?

We remark that a variation of (5.1) appears in the work of Regev [Reg10] on the number of standard Young tableaux contained in the (k,ℓ)-hook, which are tableaux whose shape is contained in

$$H(k,\ell;n) = \{\lambda = (\lambda_1, \lambda_2, \ldots) : \lambda \vdash n \text{ and } \lambda_{k+1} \leq \ell\}.$$

Specifically, it is shown in [Reg10, Proposition 3.2] that

$$\sum_{\lambda \in H^*(2,2;2m)} f^\lambda = \sum_{\lambda \in H(4,0;2m-2)} f^\lambda \tag{5.2}$$

where the partitions on the left-hand side are

$$H^*(2,2;2m) = \{(k,k,2^{m-k}) \vdash 2m : k \in \{2,\ldots,m\}\}$$

while the right-hand sum is over partitions of size $2m-2$ with up to four parts. This differs from (5.1) in that the sums in (5.2) are over partitions of differing size. We note that both sides of (5.2) are counted by the product $C_{m-1}C_m$ of Catalan numbers. This follows from the result of Gouyou-Beauchamps [GB89] that the number of standard Young tableaux of size n with at most four rows is $C_{\lfloor (n+1)/2 \rfloor} C_{\lceil (n+1)/2 \rceil}$.

Finally, we wonder if our results, especially Theorem 1.3 and Theorem 4.2, have a natural representation-theoretic interpretation. In a similar direction, can one find combinatorial proofs of our identities via bijections on the corresponding sets of standard Young tableaux? We note that even for the unrefined version of Theorem 1.3, namely Theorem 1.2, we do not know of a bijective proof. One would need a bijection between, on one side, all standard Young tableaux of shape λ where λ is a partition of n into three parts of equal parity and, on the other side, standard Young tableaux of shape $(k,k,1^{n-2k})$ where k varies. A bijection between the latter and Riordan paths is known (see, for instance, [HSW25]), and bijections are known between Young tableaux with up to three rows and Motzkin paths (see, for instance, [GMTW20] or [MV22]). It is therefore natural to look for a bijection between Motzkin paths and Young tableaux with up to three rows that sends Riordan paths to Young tableaux of shape λ where λ has parts of equal parity.

Appendix A. Proofs of some general cases of Theorem 1.3

In this section, we provide details in several illustrative general cases for the inductive proof of Theorem 1.3 outlined in Section 3. These cases depend on whether n and k have the same parity or not. In the examples below, we specify the individual parities (such as n even and k odd) but this is only to simplify the exposition. For example, whether $(n-k-1,k,1)$ is in $X_1(n,k)$ or $X_2(n,k)$ depends on the parity of k. On the other hand, reversing the parities introduces no further complications.

The first case detailed below is for k large, n even and k odd. The remaining "non boundary" cases where k is not almost equal to $n/3$ are all essentially identical: either the identity involving X_1 or X_2, but not both, require Lemma 2.3. In the remaining two cases detailed below, we therefore instead illustrate the two behaviors that occur at the boundary.

A.1. k large, n even, k odd.

In this case, we have

$$X_1(n,k) = \{(n-k-1,k,1), (n-k-3,k,3), \ldots, (k+1,k,n-2k-1)\}$$

and, since $k > \lceil \frac{n}{3} \rceil$ and $k \not\equiv n \pmod 2$, the corresponding identity to be proved is

$$\sum_{\lambda \in X_1(n,k)} f^\lambda = f^{(k+1,k,1^{n-2k-1})}. \tag{A.1}$$

Applying the branching rule to the right-hand side of (A.1) we obtain

$$f^{(k,k,1^{n-2k-1})} + f^{(k+1,k-1,1^{n-2k-1})} + f^{(k+1,k,1^{n-2k-2})}.$$

To apply the branching rule to the left-hand side of (A.1), we proceed as in Section 3 by removing a box from one row at a time. Removing a box from the first row of each diagram in $X_1(n,k)$ we get

$$\{(n-k-2,k,1),(n-k-4,k,3),\dots,(k,k,n-2k-1)\}.$$

Since both k and $n-1$ are odd, this set equals $X_1(n-1,k)$ and we obtain $f^{(k,k,1^{n-2k-1})} + f^{(k+1,k+1,1^{n-2k-3})}$ by the induction hypothesis. Likewise removing from the second row of each diagram in $X_1(n,k)$, we get

$$\{(n-k-1,k-1,1),(n-k-3,k-1,3),\dots,(k+1,k-1,n-2k-1)\}.$$

This is the set $X_1(n-1,k-1)$ but missing the term $(k-1,k-1,n-2k+1)$. By the induction hypothesis, the resulting terms equal

$$f^{(k-1,k-1,1^{n-2k+1})} + f^{(k,k,1^{n-2k-1})} - f^{(k-1,k-1,n-2k+1)}.$$

Removing from the third row results in

$$\{(n-k-1,k),(n-k-3,k,2),\dots,(k+1,k,n-2k-2)\}.$$

This is $X_2(n-1,k)$ and we get $f^{(k+1,k,1^{n-2k-2})}$ by the induction hypothesis.

Equating the results of applying the branching rule to both sides of (A.1), we find that (A.1) is equivalent to

$$\begin{aligned} &f^{(k,k,1^{n-2k-1})} + f^{(k+1,k-1,1^{n-2k-1})} + f^{(k+1,k,1^{n-2k-2})} \\ &= f^{(k,k,1^{n-2k-1})} + f^{(k+1,k+1,1^{n-2k-3})} + f^{(k-1,k-1,1^{n-2k+1})} \\ &\quad + f^{(k,k,1^{n-2k-1})} - f^{(k-1,k-1,n-2k+1)} + f^{(k+1,k,1^{n-2k-2})}. \end{aligned}$$

Cancelling identical terms simplifies this to

$$f^{(k+1,k-1,1^{n-2k-1})} = f^{(k-1,k-1,1^{n-2k+1})} + f^{(k,k,1^{n-2k-1})} + f^{(k+1,k+1,1^{n-2k-3})} - f^{(k-1,k-1,n-2k+1)},$$

which is a special case of Lemma 2.3. This completes the proof of (A.1).

On the other hand, the identity corresponding to

$$X_2(n,k) = \{(n-k,k),(n-k-2,k,2),\dots,(k,k,n-2k)\}$$

takes the form

$$\sum_{\lambda \in X_2(n,k)} f^\lambda = f^{(k,k,1^{n-2k})} + f^{(k+1,k+1,1^{n-2k-2})}. \tag{A.2}$$

Proceeding as before, removing terms from each row of the left-hand side results in equations that are known by the induction hypothesis—there are no missing terms and no need to use Lemma 2.3. To wit, removing a box from the first row of each diagram in $X_2(n,k)$ we get

$$\{(n-k-1,k),(n-k-3,k,2),\dots,(k+1,k,n-2k-2)\}.$$

Since both k and $n-1$ are odd this set is $X_2(n-1,k)$ and we obtain $f^{(k+1,k,1^{n-2k-2})}$ by the induction hypothesis. Likewise removing a box from the second row, we get

$$\{(n-k,k-1),(n-k-2,k-1,2),\dots,(k,k-1,n-2k)\}.$$

This equals the set $X_1(n-1,k-1)$ and, since $n-1$ is odd and $k-1$ is even, the inductive hypothesis gives $f^{(k,k-1,1^{n-2k})}$. Removing a box from the third row results in

$$\{(n-k-2,k,1),(n-k-4,k,3),\dots,(k,k,n-2k-1)\},$$

which is $X_1(n-1,k)$. Hence, the inductive hypothesis applies to evaluate the contribution as $f^{(k,k,1^{n-2k-1})}+f^{(k+1,k+1,1^{n-2k-3})}$. These four terms obtained from branching the left-hand side of (A.2) are the same as the four we obtain from the right-hand side, which proves the identity (A.2).

A.2. $n = 3k-1$**,** k **odd.** Here, we illustrate where the middle case of Lemma 2.3 arises, reflecting that there is no "missing" term to deal with. In the present case, we have

$$X_1(n,k)=\{(n-k-1,k,1),(n-k-3,k,3),\dots,(k+1,k,k-2)\}$$

and, since $k=\lceil n/3\rceil$, we need to show the corresponding identity

$$\sum_{\lambda\in X_1(n,k)} f^\lambda = f^{(k,k,1^{n-2k})}+f^{(k+1,k+1,1^{n-2k-2})}. \tag{A.3}$$

Branching on the right-hand side results in

$$f^{(k,k-1,1^{n-2k})}+f^{(k,k,1^{n-2k-1})}+f^{(k+1,k,1^{n-2k-2})}+f^{(k+1,k+1,1^{n-2k-3})}.$$

As before, we apply the branching rule to the left-hand side of (A.3) by removing a box from one row at a time. Removing boxes from the first row of the partitions in $X_1(n,k$, we get $\{(n-k-2,k,1),(n-k-4,k,3),\dots,(k,k,k-2)\}$ which is $X_1(n-1,k)$. Hence, the inductive hypothesis gives $f^{(k,k,1^{n-2k-1})}+f^{(k+1,k+1,1^{n-2k-3})}$. Likewise removing from the second row, we obtain $\{(n-k-1,k-1,1),(n-k-3,k-1,3),\dots,(k+1,k-1,k-2)\}$ which is just $X_2(n-1,k-1)$, so that induction gives $f^{(k,k-1,1^{n-2k})}$. Finally, removing from the third row, we get $\{(n-k-1,k),(n-k-3,k,2),\dots,(k+1,k,k-3)\}$ which is $X_2(n-1,k)$, so that induction gives $f^{(k+1,k,1^{n-2k-2})}$. Thus, applying the branching rule to the left-hand side of (A.3) results in exactly the same four terms as when applied to the right-hand side. This proves (A.3).

On the other hand, the identity corresponding to

$$X_2(n,k)=\{(n-k,k),(n-k-2,k,2),\dots,(k+2,k,k-3),(k,k,k-1)\}$$

is

$$\sum_{\lambda\in X_2(n,k)} f^\lambda = f^{(k+1,k,1^{k-2})}. \tag{A.4}$$

Branching on the right-hand side results in

$$f^{(k,k,1^{k-2})}+f^{(k+1,k-1,1^{k-2})}+f^{(k+1,k,1^{k-3})}.$$

For the left-hand side, we proceed as before: removing boxes from the first row results in $\{(n-k-1,k),(n-k-3,k,2),\dots(k+1,k,k-3)\}$ which is $X_2(n-1,k)$ and, so, the induction hypothesis gives $f^{(k+1,k,1^{k-3})}$ for the corresponding contribution. Removing from the second row, we get $\{(n-k,k-1),(n-k-2,k-1,2),\dots,(k,k-1,k-1)\}$ which is $X_1(n-1,k-1)$ so that induction gives $f^{(k-1,k-1,1^{k})}+f^{(k,k,1^{k-2})}$. Finally, removing from the third row, we get $\{(n-k-2,k,1),(n-k-4,k,3),\dots,(k,k,k-2)$ which is $X_1(n-1,k)$. Hence, induction gives $f^{(k,k,1^{k-2})}+f^{(k+1,k+1,1^{k-4})}$. Equating the right and left-hand sides and cancelling common terms, we find that we need to show that

$$f^{(k+1,k-1,1^{k-2})}=f^{(k-1,k-1,1^{k})}+f^{(k,k,1^{k-2})}+f^{(k+1,k+1,1^{k-4})}.$$

But this is precisely the equation for $L(k-1,k)$ in Lemma 2.3, thus proving (A.4).

A.3. **$n = 3x$, x even, $k = x + 1$.** In the present case, we have $k > \lceil \frac{n}{3} \rceil = x$ which means that k is "large". However, during the application of the branching rule, one of the cases requires us to use the induction hypothesis for small k. We detail the identity corresponding to

$$X_1(n, k) = \{(n - k - 1, k, 1), (n - k - 3, k, 3), \ldots, (k + 1, k, n - 2k - 1)$$

since it requires using Proposition 2.4. Because k is large and of parity opposite to n, the identity to be proved is

$$\sum_{\lambda \in X_1(n,k)} f^\lambda = f^{(k+1,k,1^{n-2k-1})}. \tag{A.5}$$

Applying the branching rule to the right-hand side results in

$$f^{(k,k,1^{n-2k-1})} + f^{(k+1,k-1,1^{n-2k-1})} + f^{(k+1,k,1^{n-2k-2})}.$$

We apply the branching rule to the left-hand side of (A.5) as in previous cases. Removing a box from the first row of each diagram in $X_1(n, k)$, we get

$$\{(n - k - 2, k, 1), (n - k - 4, k, 3), \ldots, (k + 1, k, n - 2k - 1)\},$$

which is $X_1(n-1, k)$. Since $n-1$ and k are both odd, the inductive hypothesis gives $f^{(k,k,1^{n-2k-1})} + f^{(k+1,k+1,1^{n-2k-3})}$. Removing a box from the second row of each diagram in $X_1(n, k)$, we get

$$\{(n - k - 1, k - 1, 1), (n - k - 3, k - 1, 3), \ldots, (k + 1, k - 1, n - 2k - 1)\}.$$

This is $X_2(n - 1, k - 1)$ missing the term $(k - 1, k - 1, n - 2k + 1)$. Note that, here, $k - 1$ is small compared to $n - 1$ so, even though $k - 1$ and have opposite parity, we are in the first regime of Theorem 1.3. Hence, the inductive hypothesis gives $f^{(k,k-1,1^{n-2k})} - f^{(k-1,k-1,n-2k+1)}$. Removing a box from the third row of each diagram in $X_1(n, k)$, we get

$$\{(n - k - 1, k), (n - k - 3, k, 2), \ldots, (k + 1, k, n - 2k - 2)\}.$$

This is $X_2(n - 1, k)$ which by the induction hypothesis results in $f^{(k+1,k,1^{n-2k-2})}$. Equating the left and right-hand sides and cancelling like terms, we find that we need to prove

$$f^{(k+1,k-1,1^{n-2k-1})} + f^{(k-1,k-1,n-2k+1)} = f^{(k+1,k+1,1^{n-2k-3})} + f^{(k,k-1,1^{n-2k})}. \tag{A.6}$$

By Lemma 2.3, the left-hand side of (A.6) is just $f^{(k-1,k-1,1^{n-2k+1})} + f^{(k,k,1^{n-2k-1})} + f^{(k+1,k+1,1^{n-2k-3})}$. Cancelling out the $f^{(k+1,k+1,1^{n-2k-3})}$, we are left to show that

$$f^{(k-1,k-1,1^{n-2k+1})} + f^{(k,k,1^{n-2k-1})} = f^{(k,k-1,1^{n-2k})}.$$

In the present case, we have $n - 2k + 1 = k - 2$ so that Proposition 2.4 gives the result.

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