

THE p -ADIC VALUATION OF k -CENTRAL BINOMIAL COEFFICIENTS

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ABSTRACT. The coefficients $c(n, k)$ defined by

$$(1 - k^2 x)^{-1/k} = \sum_{n \geq 0} c(n, k) x^n$$

reduce to the central binomial coefficients $\binom{2n}{n}$ for $k = 2$. Motivated by a question of H. Montgomery and H. Shapiro for the case $k = 3$, we prove that $c(n, k)$ are integers and study their divisibility properties.

1. INTRODUCTION

In a recent issue of the American Mathematical Monthly, Hugh Montgomery and Harold S. Shapiro proposed the following problem (Problem 11380, August-September 2008):

For $x \in \mathbb{R}$, let

$$(1.1) \quad \binom{x}{n} = \frac{1}{n!} \prod_{j=0}^{n-1} (x - j).$$

For $n \geq 1$, let a_n be the numerator and q_n the denominator of the rational number $\binom{-1/3}{n}$ expressed as a reduced fraction, with $q_n > 0$.

- (1) Show that q_n is a power of 3.
- (2) Show that a_n is odd if and only if n is a sum of distinct powers of 4.

Our approach to this problem employs Legendre's remarkable expression [7]:

$$(1.2) \quad \nu_p(n!) = \frac{n - s_p(n)}{p - 1},$$

that relates the p -adic valuation of factorials to the sum of digits of n in base p . For $m \in \mathbb{N}$ and a prime p , the p -adic valuation of m , denoted by $\nu_p(m)$, is the highest power of p that divides m . The expansion of $m \in \mathbb{N}$ in base p is written as

$$(1.3) \quad m = a_0 + a_1 p + \cdots + a_d p^d,$$

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with integers $0 \leq a_j \leq p-1$ and $a_d \neq 0$. The function s_p in (1.2) is defined by

$$(1.4) \quad s_p(m) := a_0 + a_1 + \cdots + a_d.$$

Since, for $n > 1$, $\nu_p(n) = \nu_p(n!) - \nu_p((n-1)!)$, it follows from (1.2) that

$$(1.5) \quad \nu_p(n) = \frac{1 + s_p(n-1) - s_p(n)}{p-1}.$$

The p -adic valuations of binomial coefficients can be expressed in terms of the function s_p :

$$(1.6) \quad \nu_p \left(\binom{n}{k} \right) = \frac{s_p(k) + s_p(n-k) - s_p(n)}{p-1}.$$

In particular, for the central binomial coefficients $C_n := \binom{2n}{n}$ and $p=2$, we have

$$(1.7) \quad \nu_2(C_n) = 2s_2(n) - s_2(2n) = s_2(n).$$

Therefore, C_n is always even and $\frac{1}{2}C_n$ is odd precisely whenever n is a power of 2. This is a well-known result.

The central binomial coefficients C_n have the generating function

$$(1.8) \quad (1-4x)^{-1/2} = \sum_{n \geq 0} C_n x^n.$$

The binomial theorem shows that the numbers in the Montgomery-Shapiro problem bear a similar generating function

$$(1.9) \quad (1-9x)^{-1/3} = \sum_{n \geq 0} \binom{-\frac{1}{3}}{n} (-9x)^n.$$

It is natural to consider the coefficients $c(n, k)$ defined by

$$(1.10) \quad (1-k^2x)^{-1/k} = \sum_{n \geq 0} c(n, k) x^n,$$

which include the central binomial coefficients as a special case. We call $c(n, k)$ the k -central binomial coefficients. The expression

$$(1.11) \quad c(n, k) = (-1)^n \binom{-\frac{1}{k}}{n} k^{2n}$$

comes directly from the binomial theorem. Thus, the Montgomery-Shapiro question from (1.1) deals with arithmetic properties of

$$(1.12) \quad \binom{-\frac{1}{3}}{n} = (-1)^n \frac{c(n, 3)}{3^{2n}}.$$

2. THE INTEGRALITY OF $c(n, k)$

It is a simple matter to verify that the coefficients $c(n, k)$ are rational numbers. The expression produced in the next proposition is then employed to prove that $c(n, k)$ are actually integers. The next section will explore divisibility properties of the integers $c(n, k)$.

Proposition 2.1. *The coefficient $c(n, k)$ is given by*

$$(2.1) \quad c(n, k) = \frac{k^n}{n!} \prod_{m=1}^{n-1} (1 + km).$$

Proof. The binomial theorem yields

$$\begin{aligned} (1 - k^2x)^{-1/k} &= \sum_{n \geq 0} \binom{-\frac{1}{k}}{n} (-k^2x)^n \\ &= \sum_{n \geq 0} \frac{k^n}{n!} \left(\prod_{m=1}^{n-1} (1 + km) \right) x^n, \end{aligned}$$

and (2.1) has been established. □

An alternative proof of the previous result is obtained from the simple recurrence

$$(2.2) \quad c(n + 1, k) = \frac{k(1 + kn)}{n + 1} c(n, k), \quad \text{for } n \geq 0,$$

and its initial condition $c(0, k) = 1$. To prove (2.2), simply differentiate (1.10) to produce

$$(2.3) \quad k(1 - k^2x)^{-1/k-1} = \sum_{n \geq 0} (n + 1) c(n + 1, k) x^n$$

and multiply both sides by $1 - k^2x$ to get the result.

Note. The coefficients $c(n, k)$ can be written in terms of the Beta function as

$$(2.4) \quad c(n, k) = \frac{k^{2n}}{nB(n, 1/k)}.$$

This expression follows directly by writing the product in (2.1) in terms of the Pochhammer symbol $(a)_n = a(a + 1) \cdots (a + n - 1)$ and the identity

$$(2.5) \quad (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)}.$$

The proof employs only the most elementary properties of the Euler's Gamma and Beta functions. The reader can find details in [1]. The conclusion is that we have an integral expression for $c(n, k)$, given by

$$(2.6) \quad c(n, k) \int_0^1 (1 - u^{1/n})^{1/k-1} du = k^{2n}.$$

It is unclear how to use it to further investigate $c(n, k)$.

In the case $k = 2$, we have that $c(n, 2) = C_n$ is a positive integer. This result extends to all values of k .

Theorem 2.2. *The coefficient $c(n, k)$ is a positive integer.*

Proof. First observe that if p is a prime dividing k , then the product in (1.10) is relatively prime to p . Therefore we need to check that $\nu_p(n!) \leq \nu_p(k^n)$. This is simple:

$$(2.7) \quad \nu_p(n!) = \frac{n - s_p(n)}{p - 1} \leq n \leq \nu_p(k^n).$$

Now let p be a prime not dividing k . Clearly,

$$(2.8) \quad \nu_p(c(n, k)) = \nu_p\left(\prod_{m < n} (1 + km)\right) - \nu_p\left(\prod_{m < n} (1 + m)\right).$$

To prove that $c(n, k)$ is an integer, we compare the p -adic valuations of $1 + km$ and $1 + m$. Observe that $1 + m$ is divisible by p^α if and only if m is of the form $\lambda p^\alpha - 1$. On the other hand, $1 + km$ is divisible by p^α precisely when m is of the form $\lambda p^\alpha - i_{p^\alpha}(k)$, where $i_{p^\alpha}(k)$ denotes the inverse of k modulo p^α in the range $1, 2, \dots, p^\alpha - 1$. Thus,

$$(2.9) \quad \nu_p(c(n, k)) = \sum_{\alpha \geq 1} \left[\frac{n + i_{p^\alpha}(k) - 1}{p^\alpha} \right] - \left[\frac{n}{p^\alpha} \right].$$

The claim now follows from $i_{p^\alpha}(k) \geq 1$. \square

Next, Theorem 2.2 will be slightly strengthened and an alternative proof be provided.

Theorem 2.3. *For $n > 0$, the coefficient $c(n, k)$ is a positive integer divisible by k .*

Proof. Expanding the right hand side of the identity

$$(2.10) \quad (1 - k^2x)^{-1} = \left((1 - k^2x)^{-1/k} \right)^k$$

by the Cauchy product formula gives

$$(2.11) \quad \sum_{i_1 + \dots + i_k = m} c(i_1, k) c(i_2, k) \cdots c(i_k, k) = k^{2m},$$

where the multisum runs through all the k -tuples of non-negative integers. Obviously $c(0, k) = 1$ and it is easy to check that $c(1, k) = k$. We proceed by induction on n , so we assume the assertion is valid for $c(1, k), c(2, k), \dots, c(n-1, k)$. We prove the same is true for $c(n, k)$. To this end, break up (2.11) as

$$(2.12) \quad kc(n, k) + \sum_{\substack{i_1 + \dots + i_k = n \\ 0 \leq i_j < n}} c(i_1, k) c(i_2, k) \cdots c(i_k, k) = k^{2n}.$$

Hence by the induction assumption $kc(n, k)$ is an integer.

To complete the proof, divide (2.12) through by k^2 and rewrite as follows

$$(2.13) \quad \frac{c(n, k)}{k} = k^{2n-2} - \frac{1}{k^2} \sum_{\substack{i_1+\dots+i_k=n \\ 0 \leq i_j < n}} c(i_1, k)c(i_2, k) \cdots c(i_k, k).$$

The key point is that each summand in (2.13) contains *at least two* terms, each one divisible by k . □

Note. W. Lang [6] has studied the numbers appearing in the generating function

$$(2.14) \quad c2(l; x) := \frac{1 - (1 - l^2x)^{1/l}}{lx},$$

that bears close relation to the case $k = -l < 0$ of equation (1.10). The special case $l = 2$ yields the Catalan numbers. The author establishes the integrality of the coefficients in the expansion of $c2$ and other related functions.

3. THE VALUATION OF $c(n, k)$

We consider now the p -adic valuation of $c(n, k)$. The special case when p divides k is easy, so we deal with it first.

Proposition 3.1. *Let p be a prime that divides k . Then*

$$(3.1) \quad \nu_p(c(n, p)) = \nu_p(k)n - \frac{n - s_p(n)}{p - 1}.$$

Proof. The p -adic valuation of $c(n, p)$ is given by

$$(3.2) \quad \nu_p(c(n, p)) = \nu_p(k)n - \nu_p(n!) = \nu_p(k)n - \frac{n - s_p(n)}{p - 1}.$$

Finally note that $s_p(n) = O(\log n)$. □

Note. For $p, k \neq 2$, we have $\nu_p(c(n, p)) \sim \left(\nu_p(k) - \frac{1}{p-1}\right)n$, as $n \rightarrow \infty$.

We now turn attention to the case where p does not divide k . Under this assumption, the congruence $kx \equiv 1 \pmod{p^\alpha}$ has a solution. Elementary arguments of p -adic analysis can be used to produce a p -adic integer that yields the inverse of k . This construction proceeds as follows: first choose b_0 in the range $\{1, 2, \dots, p-1\}$ to satisfy $kb_0 \equiv 1 \pmod{p}$. Next, choose c_1 , satisfying $kc_1 \equiv 1 \pmod{p^2}$ and write it as $c_1 = b_0 + kb_1$ with $0 \leq b_1 \leq p-1$. Proceeding in this manner, we obtain a sequence of integers $\{b_j : j \geq 0\}$, such that $0 \leq b_j \leq p-1$ and the partial sums of the *formal object* $x = b_0 + b_1p + b_2p^2 + \dots$ satisfy

$$(3.3) \quad k(b_0 + b_1p + \dots + b_{j-1}p^{j-1}) \equiv 1 \pmod{p^j}.$$

This is the standard definition of a p -adic integer and

$$(3.4) \quad i_{p^\infty}(k) = \sum_{j=0}^{\infty} b_j p^j$$

is the inverse of k in the ring of p -adic integers. The reader will find in [3] and [8] information about this topic.

Note. It is convenient to modify the notation in (3.4) and write it as

$$(3.5) \quad i_{p^\infty}(k) = 1 + \sum_{j=0}^{\infty} b_j p^j$$

which is always possible since the first coefficient cannot be zero. The reader is invited to check that, when doing so, the b_j are periodic in j with period the multiplicative order of p in $\mathbb{Z}/k\mathbb{Z}$. Furthermore, the b_j take values amongst $\lfloor p/k \rfloor, \lfloor 2p/k \rfloor, \dots, \lfloor (k-1)p/k \rfloor$. This will be exemplified in the case $k = 3$ later.

The analysis of $\nu_p(c(n, k))$ for those primes p not dividing k begins with a characterization of those indices for which $\nu_p(c(n, k)) = 0$, that is, p does not divide $c(n, k)$. The result is expressed in terms of the expansions of n in base p , written as

$$(3.6) \quad n = a_0 + a_1 p + a_2 p^2 + \dots + a_d p^d,$$

and the p -adic expansion of the inverse of k as given by (3.5).

Theorem 3.2. *Let p be a prime that does not divide k . Then $\nu_p(c(n, k)) = 0$ if and only if $a_j + b_j < p$ for all j in the range $1 \leq j \leq d$.*

Proof. It follows from (2.9) that $c(n, k)$ is not divisible by p precisely when

$$(3.7) \quad \left\lfloor \frac{1}{p^\alpha} \left(n + \sum_j b_j p^j \right) \right\rfloor = \left\lfloor \frac{n}{p^\alpha} \right\rfloor,$$

for all $\alpha \geq 1$, or equivalently, if and only if

$$(3.8) \quad \sum_{j=0}^{\alpha-1} (a_j + b_j) p^j < p^\alpha,$$

for all $\alpha \geq 1$. An inductive argument shows that this is equivalent to the condition $a_j + b_j < p$ for all j . Naturally, the a_j vanish for $j > d$, so it is sufficient to check $a_j + b_j < p$ for all $j \leq d$. \square

Corollary 3.3. *For all primes $p > k$ and $d \in \mathbb{N}$, we have $\nu_p(c(p^d, k)) = 0$.*

Proof. The coefficients of $n = p^d$ in Theorem 3.2 are $a_j = 0$ for $0 \leq j \leq d-1$ and $a_d = 1$. Therefore the restrictions on the coefficients b_j become $b_j < p$

for $0 \leq j \leq d-1$ and $b_d < p-1$. It turns out that $b_j \neq p-1$ for all $j \in \mathbb{N}$. Otherwise, for some $r \in \mathbb{N}$, we have $b_r = p-1$ and the equation

$$(3.9) \quad k \left(1 + \sum_{j=0}^{r-1} b_j p^j + b_r p^r \right) \equiv k \left(1 + \sum_{j=0}^{r-1} b_j p^j - p^r \right) \equiv 1 \pmod{p^{r+1}},$$

is impossible in view of

$$(3.10) \quad -kp^r < k \left(1 + \sum_{j=0}^{r-1} b_j p^j - p^r \right) < 0.$$

□

Now we return again to the Montgomery-Shapiro question. The identity (1.12) shows that the denominator q_n is a power of 3. We now consider the indices n for which $c(n, 3)$ is odd and provide a proof of the second part of their problem.

Corollary 3.4. *The coefficient $c(n, 3)$ is odd precisely when n is a sum of distinct powers of 4.*

Proof. The result follows from Theorem 3.2 and the explicit formula

$$(3.11) \quad i_{2^\infty}(3) = 1 + \sum_{j=0}^{\infty} 2^{2j+1},$$

for the inverse of 3, so that $b_{2j} = 0$ and $b_{2j+1} = 1$. Therefore, if $c(n, 3)$ is odd, the theorem now shows that $a_j = 0$ for j odd, as claimed. □

More generally, the discussion of $\nu_p(c(n, 3)) = 0$ is divided according to the residue of p modulo 3. This division is a consequence of the fact that for $p = 3u + 1$, we have

$$(3.12) \quad i_{p^\infty}(3) = 1 + 2u \sum_{m=0}^{\infty} p^m,$$

and for $p = 3u + 2$, one computes $p^2 = 3(3u^2 + 4u + 1) + 1$, to conclude that

$$(3.13) \quad i_{p^\infty}(3) = 1 + 2(3u^2 + 4u + 1) \sum_{m=0}^{\infty} p^{2m} = 1 + \sum_{m=0}^{\infty} up^{2m} + (2u+1)p^{2m+1}.$$

Theorem 3.5. *Let $p \neq 3$ be a prime and $n = a_0 + a_1p + a_2p^2 + \dots + a_dp^d$ as before. Then p does not divide $c(n, 3)$ if and only if the p -adic digits of n satisfy*

$$(3.14) \quad a_j < \begin{cases} p/3 & \text{if } j \text{ is odd or } p = 3u + 1, \\ 2p/3 & \text{otherwise.} \end{cases}$$

For general k we have the following analogous statement.

Theorem 3.6. *Let $p = ku + 1$ be a prime. Then p does not divide $c(n, k)$ if and only if the p -adic digits of n are less than p/k .*

Observe that Theorem 3.6 implies the following well-known property of the central binomial coefficients: C_n is not divisible by $p \neq 2$ if and only if the p -adic digits of n are less than $p/2$.

Now we return to (2.9) which will be written as

$$(3.15) \quad \nu_p(c(n, k)) = \sum_{\alpha \geq 0} \left\lfloor \frac{1}{p^{\alpha+1}} \sum_{m=0}^{\alpha} (a_m + b_m)p^m \right\rfloor.$$

From here, we bound

$$(3.16) \quad \sum_{m=0}^{\alpha} (a_m + b_m)p^m \leq \sum_{m=0}^{\alpha} (2p - 2)p^m = 2(p^{\alpha+1} - 1) < 2p^{\alpha+1}.$$

Therefore, each summand in (3.15) is either 0 or 1. The p -adic valuation of $c(n, p)$ counts the number of 1's in this sum. This proves the final result.

Theorem 3.7. *Let p be a prime that does not divide k . Then, with the previous notation for a_m and b_m , we have that $\nu_p(c(n, k))$ is the number of indices m such that either*

- $a_m + b_m \geq p$ or
- there is $j \leq m$ such that $a_{m-i} + b_{m-i} = p - 1$ for $0 \leq i \leq j - 1$ and $a_{m-j} + b_{m-j} \geq p$.

Corollary 3.8. *Let p be a prime that does not divide k , and write $n = \sum a_m p^m$ and $i_{p^\infty}(k) = 1 + \sum b_m p^m$, as before. Let v_1 and v_2 be the number of indices m such that $a_m + b_m \geq p$ and $a_m + b_m \geq p - 1$, respectively. Then*

$$(3.17) \quad v_1 \leq \nu_p(c(n, k)) \leq v_2.$$

4. A q -GENERALIZATION OF $c(n, k)$

A standard procedure to generalize an integer expression is to replace $n \in \mathbb{N}$ by the polynomial

$$(4.1) \quad [q]_n := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}.$$

The original expression is recovered as the limiting case $q \rightarrow 1$. For example, the factorial $n!$ is extended to the polynomial

$$(4.2) \quad [n]_q! := [n]_q [n-1]_q \dots [2]_q [1]_q = \prod_{j=1}^n \frac{1 - q^j}{1 - q}.$$

The reader will find in [5] an introduction to this q -world.

In this spirit we generalize the integers

$$(4.3) \quad c(n, k) = \frac{k^n}{n!} \prod_{m=0}^{n-1} (km + 1) = \prod_{m=1}^n \frac{k(k(m-1) + 1)}{m},$$

into the q -world as

$$(4.4) \quad F_{n,k}(q) := \prod_{m=1}^n \frac{[km]_q [k(m-1) + 1]_q}{[m]_q^2}.$$

Note that this expression indeed gives $c(n, k)$ as $q \rightarrow 1$. The corresponding extension of Theorem 2.2 is stated in the next result. The proof is similar to that given above, so it is left to the curious reader.

Theorem 4.1. *The function*

$$(4.5) \quad F_{n,k}(q) := \prod_{m=1}^n \frac{(1 - q^{km})(1 - q^{k(m-1)+1})}{(1 - q^m)^2}$$

is a polynomial in q with integer coefficients.

5. FUTURE DIRECTIONS

In this final section we discuss some questions related to the integers $c(n, k)$.

• **A combinatorial interpretation.** The integers $c(n, 2)$ are given by the central binomial coefficients $C_n = \binom{2n}{n}$. These coefficients appear in many counting situations: C_n gives the number of walks of length $2n$ on an infinite linear lattice that begin and end at the origin. Moreover, they provide the exact answer for the elementary sum

$$(5.1) \quad \sum_{k=0}^n \binom{n}{k}^2 = C_n.$$

Is it possible to produce similar results for $c(n, k)$, with $k \neq 2$? In particular, what do the numbers $c(n, k)$ count?

• **A further generalization.** The polynomial $F_{n,k}(q)$ can be written as

$$(5.2) \quad F_{n,k}(q) = \frac{(1 - q)}{(1 - q^{kn+1})} \prod_{m=1}^n \frac{(1 - q^{km})(1 - q^{k(m+1)})}{(1 - q^m)^2}$$

which suggests the extension

$$(5.3) \quad G_{n,k}(q, t) := \frac{(1 - q)}{(1 - tq^{kn})} \prod_{m=1}^n \frac{(1 - q^{km})(1 - tq^{km})}{(1 - q^m)^2}$$

so that $F_{n,k}(q) = G_{n,k}(q, q)$. Observe that $G_{n,k}(q, t)$ is not always a polynomial. For example,

$$(5.4) \quad G_{2,1}(q, t) = \frac{1 - qt}{1 - q^2}.$$

On the other hand,

$$(5.5) \quad G_{1,2}(q, t) = q + 1.$$

The following functional equation is easy to establish.

Proposition 5.1. *The function $G_{n,k}(q, t)$ satisfies*

$$(5.6) \quad G_{n,k}(q, tq^k) = \frac{(1 - q^{kn}t)}{(1 - q^kt)} G_{n,k}(q, t).$$

The reader is invited to explore its properties. In particular, find minimal conditions on n and k to guarantee that $G_{n,k}(q, t)$ is a polynomial.

Consider now the function

$$(5.7) \quad H_{n,k,j}(q) := G_{n,k}(q, q^j)$$

that extends $F_{n,k}(q) = H_{n,k,1}(q)$. The following statement predicts the situation where $H_{n,k,j}(q)$ is a polynomial.

Problem. Show that $H_{n,k,j}(q)$ is a polynomial precisely if the indices satisfy $k \equiv 0 \pmod{\gcd(n, j)}$.

• **A result of Erdős, Graham, Ruzsa and Strauss.** In this paper we have explored the conditions on n that result in $\nu_p(c(n, k)) = 0$. Given two distinct primes p and q , P. Erdős et al. [2] discuss the existence of indices n for which $\nu_p(C_n) = \nu_q(C_n) = 0$. Recall that by Theorem 3.6 such numbers n are characterized by having p -adic digits less than $p/2$ and q -adic digits less than $q/2$. The following result of [2] proves the existence of infinitely many such n .

Theorem 5.2. *Let $A, B \in \mathbb{N}$ such that $A/(p-1) + B/(q-1) \geq 1$. Then there exist infinitely many numbers n with p -adic digits $\leq A$ and q -adic digits $\leq B$.*

This leaves open the question for $k > 2$ whether or not there exist infinitely many numbers n such that $c(n, k)$ is neither divisible by p nor by q . The extension to more than two primes is open even in the case $k = 2$. In particular, a prize of \$1000 has been offered by R. Graham for just showing that there are infinitely many n such that C_n is coprime to $105 = 3 \cdot 5 \cdot 7$. On the other hand, it is conjectured that there are only finitely many indices n such that C_n is not divisible by any of 3, 5, 7 and 11.

Finally, we remark that Erdős et al. conjectured in [2] that the central binomial coefficients C_n are never squarefree for $n > 4$ which has been proved

by Granville and Ramare in [4]. Define

$$(5.8) \quad \tilde{c}(n, k) := \text{Numerator} (k^{-n}c(n, k)) .$$

We have *some* empirical evidence which suggests the existence of an index $n_0(k)$, such that $\tilde{c}(n, k)$ is not squarefree for $n \geq n_0(k)$. The value of $n_0(k)$ could be large. For instance

$$\begin{aligned} \tilde{c}(178, 5) = & 10233168474238806048538224953529562250076040177895261 \\ & 58561031939088200683714293748693318575050979745244814 \\ & 765545543340634517536617935393944411414694781142 \end{aligned}$$

is squarefree, so that $n_0(5) \geq 178$. The numbers $\tilde{c}(n, k)$ present new challenges, even in the case $k = 2$. Recall that $\frac{1}{2}C_n$ is odd if and only if n is a power of 2. Therefore, C_{786} is not squarefree. On the other hand, the complete factorization of C_{786} shows that $\tilde{c}(786, 2)$ is squarefree. We conclude that $n_0(2) \geq 786$.

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