

IDENTITIES FOR BERNOULLI POLYNOMIALS RELATED TO MULTIPLE TORNHEIM ZETA FUNCTIONS

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ABSTRACT. We show that each member of a doubly infinite sequence of highly nonlinear expressions of Bernoulli polynomials, which can be seen as linear combinations of certain higher-order convolutions, is a multiple of a specific product of linear factors. The special case of Bernoulli numbers has important applications in the study of multiple Tornheim zeta functions. The proof of the main result relies on properties of Eulerian polynomials and higher-order Bernoulli polynomials.

1. INTRODUCTION

Various convolution identities for Bernoulli polynomials and related numbers and polynomials have attracted considerable attention in recent years. In connection with a detailed study of multiple Tornheim zeta functions, the first author [4, 5] recently obtained what appears to be a new type of identity, namely

$$(1.1) \quad \sum_{m=1}^n \binom{n+1}{m} \sum_{\substack{j_1, \dots, j_m \geq 1 \\ j_1 + \dots + j_m = n}} \prod_{i=1}^m \frac{B_{j_i}(z)}{j_i!} = \frac{1}{n!} \prod_{j=1}^n ((n+1)z - j),$$

for $n \geq 1$. Here $B_k(z)$ is the k th *Bernoulli polynomial*, which can be defined by the generating function

$$(1.2) \quad \frac{x e^{zx}}{e^x - 1} = \sum_{k=0}^{\infty} B_k(z) \frac{x^k}{k!}, \quad |x| < 2\pi,$$

and the k th *Bernoulli number* is defined by $B_k := B_k(0)$, $k \geq 0$. The first few Bernoulli polynomials are listed in Table 2 in Section 2 below.

A notable feature of the identity (1.1) is the fact that an easy linear combination of convolutions results in a product of n monomials. When the Bernoulli polynomials on the left are replaced by the corresponding Bernoulli *numbers*, i.e., setting $z = 0$, then the right-hand side is simply $(-1)^n$.

Also useful in connection with multiple Tornheim zeta functions is an identity that involves a generalization of the left-hand side of (1.1). For all integers $n \geq 1$

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and $k \geq 0$, we define

$$(1.3) \quad S_{n,k}(z) := \sum_{m=1}^n \binom{n+1}{m} k!^{n-m} (-1)^{km} \sum_{\substack{j_1, \dots, j_m \geq 0 \\ j_1 + \dots + j_m = (k+1)(n-m)+k}} \prod_{i=1}^m \frac{B_{k+1+j_i}(z)}{j_i!(k+1+j_i)}.$$

When $k = 0$, a simple shift in indexing shows that $S_{n,0}(z)$ is the left-hand side of (1.1). The methods used in [4] for $k = 0$ do not appear to apply when $k \geq 1$. It is the purpose of this paper to deal with this case. To motivate our main result, we list the first few cases of $S_{n,1}(z)$ and $S_{n,2}(z)$ in Table 1.

k	n	$S_{n,k}(z)$
1	1	$\frac{-1}{3}z(z-1)(2z-1)$
1	2	$\frac{1}{20}z(z-1)(3z-1)(3z-2)(2z-1)$
1	3	$\frac{-1}{105}z(z-1)(4z-1)(2z-1)(4z-3)(z^2-z+1)$
1	4	$\frac{-1}{18144}z(z-1)(5z-1)(5z-2)(5z-3)(5z-4)(2z-1)(13z^2-13z-6)$
2	1	$\frac{1}{30}z(z-1)(2z-1)(3z^2-3z-1)$
2	2	$\frac{1}{160}z^2(z-1)^2(3z-1)(3z-2)(7z^2-7z-2)$
2	3	$\frac{1}{20790}z(z-1)(4z-1)(2z-1)(4z-3)(321z^6 - \dots - 3)$
2	4	$\frac{1}{16765056}z^2(z-1)^2(5z-1) \cdots (5z-4)(19302z^6 - \dots - 348)$

Table 1: $S_{n,1}(z)$ and $S_{n,2}(z)$ for $1 \leq n \leq 4$.

We see that for both $k = 1$ and $k = 2$, the polynomial $S_{n,k}(z)$ is divisible by $z((n+1)z-1) \cdots ((n+1)z-(n+1))$. It is the main purpose of this paper to prove that this observation is in fact true for all n and k .

Theorem 1. *For all integers $n \geq 1$ and $k \geq 1$, the polynomial $S_{n,k}(z)$ satisfies*

$$(1.4) \quad S_{n,k}(1-z) = (-1)^{(k+1)(n+1)-1} S_{n,k}(z),$$

and is divisible by

$$z \prod_{j=1}^{n+1} ((n+1)z - j).$$

As an immediate consequence of Theorem 1 we obtain a corresponding statement about Bernoulli numbers, which we can phrase as follows.

Corollary 2. *For all integers $n \geq 1$ and $k \geq 1$ we have $S_{n,k}(0) = S_{n,k}(1) = 0$, and when at least one of k and n is odd, then $S_{n,k}(\frac{1}{2}) = 0$.*

In particular, since $B_m(0) = B_m$, this means that the right-hand side of (1.3), with Bernoulli polynomials replaced by Bernoulli numbers, is 0 for all $n \geq 1$ and $k \geq 1$.

In order to prove Theorem 1, we define an auxiliary power series in Section 2 and prove some lemmas involving this series. In Section 3 we complete the proof of Theorem 1. Section 4 contains a few further results, and we conclude this paper with some additional remarks in Section 5.

2. SOME LEMMAS

We introduce a specific power series as an auxiliary function. For any integer $k \geq 0$ and complex variable z we define

$$(2.1) \quad F_k(x, z) := 1 + (-1)^k \frac{x^{k+1}}{k!} \sum_{m=0}^{\infty} \frac{B_{m+k+1}(z)}{m+k+1} \frac{x^m}{m!}.$$

This series has the same radius of convergence, 2π , as (1.2). Using the reflection identity for Bernoulli polynomials, namely $B_n(1-x) = (-1)^n B_n(x)$ (see, e.g., [11, 24.4.3]), it is easy to verify that

$$(2.2) \quad F_k(-x, 1-z) = F_k(x, z).$$

The function $F_k(x, 0)$, i.e., the case where on the right of (2.1) we have Bernoulli *numbers* instead of polynomials, has previously been studied and applied by several authors; see [2], [6], and [12].

In what follows we denote the coefficient of x^k , $k \geq 0$, in a power series $f(x)$ by $[x^k]f(x)$. We can now state and prove the following result.

Lemma 3. *For all $n \geq 1$ and $k \geq 0$ we have*

$$(2.3) \quad \frac{S_{n-1,k}(z)}{k!^{n-1}} = [x^{(k+1)n-1}]F_k(x, z)^n.$$

Proof. We rewrite (1.3) as

$$(2.4) \quad \frac{S_{n,k}(z)}{k!^n} = \sum_{m=1}^n \binom{n+1}{m} S_{n,k,m}(z) = \sum_{m=0}^{n+1} \binom{n+1}{m} S_{n,k,m}(z),$$

where

$$S_{n,k,m}(z) := \sum_{\substack{j_1, \dots, j_m \geq 1 \\ j_1 + \dots + j_m = k(n+1-m) + n}} \prod_{i=1}^m \frac{(-1)^k B_{j_i+k}(z)}{k!(j_i-1)!(j_i+k)}.$$

We now define

$$(2.5) \quad G_k(x, z) := \frac{(-1)^k}{k!} \sum_{j=1}^{\infty} \frac{B_{j+k}(z)}{(j-1)!(j+k)} x^j$$

and first observe that

$$(2.6) \quad S_{n,k,m}(z) = [x^{k(n+1-m)+n}]G_k(x, z)^m.$$

Then Cauchy's integral formula, applied to the right of (2.6), gives

$$S_{n,k,m}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{G_k(x, z)^m}{x^{k(n+1-m)+n+1}} dx,$$

where the contour γ traverses, for instance, a circle around the origin with small enough radius, once in the positive direction. Then, after replacing n by $n-1$, we

get with (2.4),

$$\begin{aligned}
\frac{S_{n-1,k}(z)}{k!^{n-1}} &= \frac{1}{2\pi i} \sum_{m=0}^n \binom{n}{m} \int_{\gamma} \frac{G_k(x,z)^m}{x^{k(n-m)+n}} dx \\
&= \frac{1}{2\pi i} \int_{\gamma} \frac{1}{x^{(k+1)n}} \sum_{m=0}^n \binom{n}{m} x^{km} G_k(x,z)^m dx \\
&= \frac{1}{2\pi i} \int_{\gamma} \frac{(1+x^k G_k(x,z))^n}{x^{(k+1)n}} dx \\
&= [x^{(k+1)n-1}] (1+x^k G_k(x,z))^n,
\end{aligned}$$

where we have used Cauchy's integral formula again. Finally, since by (2.1) and (2.5) we have

$$1 + x^k G_k(x, z) = F_k(x, z),$$

this proves our lemma. \square

In what follows, we require the *Eulerian polynomials* which can be defined by the generating function

$$(2.7) \quad \frac{1-y}{1-ye^{(1-y)t}} = \sum_{k=0}^{\infty} A_k(y) \frac{t^k}{k!};$$

see, e.g., [3, p. 244]. The first few Eulerian polynomials are listed in Table 2.

k	$B_k(z)$	$A_k(y)$
0	1	1
1	$z - \frac{1}{2}$	y
2	$z^2 - z + \frac{1}{6}$	$y^2 + y$
3	$z^3 - \frac{3}{2}z^2 + \frac{1}{2}z$	$y^3 + 4y^2 + y$
4	$z^4 - 2z^3 + z^2 - \frac{1}{30}$	$y^4 + 11y^3 + 11y^2 + y$
5	$z^5 - \frac{5}{2}z^4 + \frac{5}{3}z^3 - \frac{1}{6}z$	$y^5 + 26y^4 + 66y^3 + 26y^2 + y$
6	$z^6 - 3z^5 + \frac{5}{2}z^4 - \frac{1}{2}z^2 + \frac{1}{42}$	$y^6 + 57y^5 + 302y^4 + 302y^3 + 57y^2 + y$

Table 2: $B_k(z)$ and $A_k(y)$ for $0 \leq k \leq 6$.

The Eulerian polynomials are self-reciprocal (or palindromic), and we can write

$$(2.8) \quad A_k(y) = \sum_{j=0}^k A(k, j) y^j \quad (k \geq 0).$$

The coefficients $A(k, j)$ are the well-known *Eulerian numbers*, which have important combinatorial interpretations. We also require the generating function

$$(2.9) \quad \frac{A_k(y)}{(1-y)^{k+1}} = \sum_{m=0}^{\infty} m^k y^m;$$

see, e.g., [3, p. 245]. We can now rewrite the sequence of functions $F_k(x, z)$ that was defined in (2.1).

Lemma 4. For all $k \geq 1$ we have

$$(2.10) \quad F_k(x, z) = \left(\frac{x}{e^x - 1} \right)^{k+1} e^{zx} \sum_{j=0}^k (1 - e^x)^j \frac{A_{k-j}(e^x) z^j}{(k-j)! j!}.$$

Proof. Applying a variant of a method used in [6], we find with (1.2),

$$(2.11) \quad \begin{aligned} \sum_{m=1}^{\infty} e^{(z-m)x} &= e^{zx} \frac{e^{-x}}{1 - e^{-x}} = \frac{e^{zx}}{e^x - 1} \\ &= \frac{1}{x} + \frac{1}{x} \left(\sum_{m=0}^{\infty} B_m(z) \frac{x^m}{m!} - 1 \right) \\ &= \frac{1}{x} + \sum_{m=0}^{\infty} \frac{B_{m+1}(z)}{m+1} \cdot \frac{x^m}{m!}. \end{aligned}$$

Note that the expressions in (2.11) have simple poles at $x = 0$. Taking the k th derivative with respect to x of both sides of (2.11), we get

$$\sum_{m=1}^{\infty} (z-m)^k e^{(z-m)x} = (-1)^k \frac{k!}{x^{k+1}} + \sum_{m=0}^{\infty} \frac{B_{m+k+1}(z)}{m+k+1} \cdot \frac{x^m}{m!}.$$

Comparing this with (2.1), we immediately see that

$$(2.12) \quad F_k(x, z) = \frac{x^{k+1}}{k!} \sum_{m=1}^{\infty} (m-z)^k e^{(z-m)x}.$$

Expanding $(m-z)^k$ as a binomial sum and changing the order of summation, we get

$$(2.13) \quad F_k(x, z) = x^{k+1} e^{zx} \sum_{j=0}^k \frac{(-z)^j}{j!(k-j)!} \sum_{m=1}^{\infty} m^{k-j} (e^{-x})^m.$$

Now by (2.9) and the palindromic property of the polynomials $A_k(y)$ we have for $0 \leq j \leq k-1$,

$$\begin{aligned} \sum_{m=1}^{\infty} m^{k-j} (e^{-x})^m &= \frac{A_{k-j}(e^{-x})}{(1 - e^{-x})^{k-j+1}} = \frac{e^{-x(k-j+1)} A_{k-j}(e^x)}{(1 - e^{-x})^{k-j+1}} \\ &= \frac{(e^x - 1)^j}{(e^x - 1)^{k+1}} A_{k-j}(e^x), \end{aligned}$$

while for $j = k$ we use the first line of (2.11) with $z = 0$, obtaining

$$\sum_{m=1}^{\infty} (e^{-x})^m = \frac{1}{e^x - 1} = \frac{(e^x - 1)^k}{(e^x - 1)^{k+1}} A_0(e^x).$$

If we combine these last two identities with (2.13), we immediately get (2.10). \square

Remarks. (1) An important special function, the *polylogarithm*, is defined by

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s},$$

which for a fixed $s \in \mathbb{C}$ defines an analytic function of z for $|z| < 1$; see, e.g., [11, 25.12.10]. Comparing this with (2.9), we get

$$A_k(y) = (1 - y)^{k+1} Li_{-k}(y),$$

and if we set $y = e^x$ and replace k by $k - j$, we see that (2.10) simplifies to

$$(2.14) \quad F_k(x, z) = (-x)^{k+1} e^{xz} \sum_{j=0}^k \frac{Li_{j-k}(e^x) z^j}{(k-j)! j!}.$$

Note that the terms $e^x - 1$ in (2.10) have disappeared.

(2) For an alternative approach to Lemma 4, see Part 4 of Section 5.

3. PROOF OF THEOREM 1

By Lemma 3 we need to determine the coefficient of $x^{(k+1)n-1}$ in $F_k(x, z)^n$. To do so, we first change the order of summation in (2.10) and obtain

$$(3.1) \quad F_k(x, z) = \left(\frac{x}{e^x - 1} \right)^{k+1} e^{xz} (z(1 - e^x))^k \sum_{j=0}^k \frac{A_j(e^x)}{(k-j)!} \cdot \frac{(z(1 - e^x))^{-j}}{j!}.$$

Taking the n th power of the sum in (3.1) gives

$$\sum_{\nu \geq 0} (z(1 - e^x))^{-\nu} \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = \nu}} \frac{A_{j_1}(e^x)}{(k-j_1)! j_1!} \cdots \frac{A_{j_n}(e^x)}{(k-j_n)! j_n!},$$

and the n th power of (3.1) then becomes

$$(3.2) \quad F_k(x, z)^n = \left(\frac{x}{e^x - 1} \right)^{n(k+1)} \frac{e^{n x z}}{(k!)^n} \sum_{\nu \geq 0} (z(1 - e^x))^{n k - \nu} \\ \times \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = \nu}} \binom{k}{j_1} A_{j_1}(e^x) \cdots \binom{k}{j_n} A_{j_n}(e^x).$$

Now let $S_{k,\nu}^{(n)}(e^x)$ be the multiple sum in (3.2). It is not difficult to compute the first few of these sums:

$$S_{k,0}^{(n)}(e^x) = 1,$$

$$S_{k,1}^{(n)}(e^x) = \frac{nk}{1!} e^x,$$

$$S_{k,2}^{(n)}(e^x) = \frac{nk}{2!} e^x [(k-1) + (kn-1)e^x],$$

$$S_{k,3}^{(n)}(e^x) = \frac{nk}{3!} e^x [(k-1)(k-2) + (k-1)(3kn+k-8)e^x + (kn-1)(kn-2)e^{2x}].$$

These polynomials will be further investigated in Section 4.

Proof of Theorem 1. The identity (1.4) follows from Lemma 3 and (2.2), with n replaced by $n+1$.

For the remainder of the proof we rewrite (3.2) as

$$(3.3) \quad F_k(x, z)^n = \left(\frac{x}{e^x - 1} \right)^{n(k+1)} \frac{e^{n x z}}{(k!)^n} \sum_{\nu=0}^{nk} (z(1 - e^x))^{n k - \nu} S_{k,\nu}^{(n)}(e^x).$$

We first consider the sum over ν . For $\nu = 0$, the corresponding summand is simply

$$(3.4) \quad z^{nk} (1 - nke^x + \cdots + (-1)^{nk} e^{nkx}),$$

while for $1 \leq \nu \leq nk$, the summand is

$$(3.5) \quad z^{n(k+1)} (1 - e^x)^{nk-\nu} S_{k,\nu}^{(n)}(e^x) = z^{n(k+1)} \sum_{j=1}^{nk} d_j^{(\nu)} e^{jx},$$

for certain coefficients $d_j^{(\nu)}$. Altogether, this and (3.4), combined with (3.3), gives

$$(3.6) \quad \begin{aligned} F_k(x, z)^n &= \left(\frac{x}{e^x - 1} \right)^{n(k+1)} \frac{e^{nxz}}{(k!)^n} \sum_{\nu=0}^{nk} z^{nk-\nu} \sum_{j=1}^{nk} d_j^{(\nu)} e^{jx} \\ &= \frac{1}{(k!)^n} \sum_{\nu=0}^{nk} z^\nu \sum_{j=0}^{nk} d_j^{(nk-\nu)} \left[\left(\frac{x}{e^x - 1} \right)^{n(k+1)} e^{(nz+j)x} \right], \end{aligned}$$

where $d_0^{(0)} = 1$ and $d_0^{(\nu)} = 0$ for $1 \leq \nu \leq nk$.

At this point we use the Bernoulli polynomials of order r , defined by the generating function

$$(3.7) \quad \left(\frac{x}{e^x - 1} \right)^r e^{zx} = \sum_{m=0}^{\infty} B_m^{(r)}(z) \frac{x^m}{m!}, \quad |x| < 2\pi;$$

see, e.g., [10, p. 127]. Usually r is a positive integer, but it can also be considered as a variable. In particular, comparing (3.7) with (1.2) shows that $B_m^{(1)}(z) = B_m(z)$.

The right-most term of (3.6) now leads to

$$(3.8) \quad [x^{n(k+1)-1}] F_k(x, z)^n = \frac{1}{(k!)^n (n(k+1) - 1)!} \sum_{\nu=0}^{nk} z^\nu \sum_{j=0}^{nk} d_j^{(nk-\nu)} B_{n(k+1)-1}^{(n(k+1))}(nz + j).$$

Using (2.3) and the fact that

$$(3.9) \quad B_{m-1}^{(m)}(y) = (y-1)(y-2) \cdots (y-m+1)$$

(see, e.g., [10, p. 130]), we get

$$(3.10) \quad S_{n-1,k}(z) = \frac{1}{k!(n(k+1) - 1)!} \sum_{\nu=0}^{nk} z^\nu \sum_{j=0}^{nk} d_j^{(nk-\nu)} \prod_{r=1}^{n(k+1)-1} (nz + j - r).$$

The product on the right is always divisible by z when $j \geq 1$. When $j = 0$, then $d_j^{(nk-\nu)} = 0$ for all $\nu < nk$, while for $\nu = nk$ we have the factor z^{nk} in the corresponding summand on the right of (3.10). Altogether, $S_{n-1,k}(z)$ is therefore divisible by z , and by (1.4) is also divisible by $z - 1$.

Finally, we consider again the product on the right-hand side of (3.10), for $j = 0, 1, \dots, nk$, and display it as follows:

$$\begin{aligned} j = 0 : & \quad (nz - 1)(nz - 2) \cdots (nz - n(k+1) + 1), \\ j = 1 : & \quad nz(nz - 1) \cdots (nz - n(k+1) + 2), \\ & \quad \vdots \\ j = nk : & \quad (nz + nk - 1)(nz + nk - 2) \cdots (nz - n + 1). \end{aligned}$$

We now see that each of the factors $nz - 1, nz - 2, \dots, nz - n + 1$ occurs in all $nk + 1$ products. This means that, by (3.10), the polynomial $S_{n-1,k}(z)$ is divisible by the product of these $n - 1$ terms. Replacing, finally, n by $n + 1$, this completes the proof of Theorem 1. \square

4. SOME FURTHER RESULTS

We return to the polynomials $S_{k,\nu}^{(n)}(y)$, defined as part of the identity (3.2) by

$$(4.1) \quad S_{k,\nu}^{(n)}(y) = \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = \nu}} \binom{k}{j_1} A_{j_1}(y) \cdots \binom{k}{j_n} A_{j_n}(y).$$

Some of the coefficients of these polynomials are relatively easy to determine.

Lemma 5. *For $\nu \geq 1$ we have*

$$S_{k,\nu}^{(n)}(y) = \sum_{j=1}^{\nu} c_j y^j,$$

where

$$c_1 = n \binom{k}{\nu}, \quad c_2 = \binom{n}{2} \left[\binom{2k}{\nu} - 2 \binom{k}{\nu} \right] + n(2^\nu - \nu - 1) \binom{k}{\nu}, \quad c_\nu = \binom{nk}{\nu}.$$

Proof. By the known properties of the Eulerian polynomials, the lowest power of y in $A_k(y)$ for all $k \geq 1$ is y^1 , and the coefficient is always 1. Hence we get a contribution to the coefficient of y in $S_{k,\nu}^{(n)}(y)$ if and only if all but one of the summation indices j_i are 0, and one has to be ν . Since there are n such cases, the coefficient of y is $n \binom{k}{\nu}$, as claimed.

Next, to determine the coefficient c_2 , we need to consider two possibilities for the defining sum (4.1). First, we assume that all but two of the j_i are 0, say $j_3 = \dots = j_n = 0$, while $j_1 \geq 1$ and $j_2 \geq 1$. Then $j_1 + j_2 = \nu$, and thus $1 \leq j_1, j_2 \leq \nu - 1$. Also, the coefficients of y in $A_{j_i}(y)$ are 1. Hence with this assumption the sum becomes

$$\sum_{r=1}^{\nu-1} \binom{k}{r} \binom{k}{\nu-r} = \binom{2k}{\nu} - 2 \binom{k}{\nu},$$

where we have used the Chu-Vandermonde convolution; see, e.g., [13, p. 8]. Since there are $\binom{n}{2}$ ways of selecting two nonzero j_i , the total contribution is

$$(4.2) \quad \binom{n}{2} \left[\binom{2k}{\nu} - 2 \binom{k}{\nu} \right].$$

Second, we assume that all but one j_i are 0. In this case the contribution is $n \binom{k}{\nu}$ times the coefficient of y^2 in $A_\nu(y)$, namely $A(\nu, 2) = 2^\nu - \nu - 1$, where this evaluation of the Eulerian number can be found, e.g., in [3, p. 243]. This, combined with (4.2), gives the coefficient c_2 .

Finally, since all $A_j(y)$ have degree j and leading coefficient 1, and since $j_1 + \dots + j_n = \nu$, the polynomial $S_{k,\nu}^{(n)}(y)$ has degree ν and leading coefficient

$$\sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = \nu}} \binom{k}{j_1} \cdots \binom{k}{j_n} = \binom{nk}{\nu},$$

where we have used the generalized Vandermonde identity. This completes the proof of the lemma. \square

We note that Lemma 5 is consistent with the four examples given after (3.2). Another easy property of the polynomials $S_{k,\nu}^{(n)}(y)$ will be required in the proof of the next theorem.

Lemma 6. *For positive integers n and k we have*

$$(4.3) \quad S_{k,nk}^{(n)}(y) = A_k(y)^n \quad \text{and} \quad y^n \mid S_{k,nk}^{(n)}(y),$$

where $A_k(y)$ is the k th Eulerian polynomial defined by (2.7).

Proof. For $\nu = nk$, the only summand in the defining multiple sum for $S_{k,\nu}^{(n)}(y)$ that does not vanish corresponds to $j_1 = \cdots = j_n = k$, which implies the first part of (4.3). The second part follows from the fact that $y \mid A_k(y)$ for all $k \geq 1$. \square

To motivate the following result, we note that Table 1 seems to indicate that the polynomial $S_{n,k}(z)$, defined in (1.3), is divisible by $z^2(z-1)^2$ when $k=2$ and n is even. This is in fact true in general.

Theorem 7. *If n, k are positive even integers, then $S_{n,k}(z)$ is divisible by $z^2(z-1)^2$.*

Proof. By (3.10) we only need to consider the summand for $\nu = 0$, and we are done if we can show that

$$(4.4) \quad T_{n,k}(z) := \sum_{j=0}^{nk} d_j^{(nk)} \prod_{r=1}^{n(k+1)-1} (nz + j - r)$$

is divisible by z^2 when k is even and n is odd (note the shift in n in (3.10)). By (3.5) with $\nu = nk$ and by (4.3) we have $d_j^{(nk)} = 0$ for $0 \leq j \leq n-1$, while $a_j := d_{n+j}^{(nk)}$, $j = 0, 1, \dots, nk-n$ are the coefficients of $(A_k(y)/y)^n$, which is a polynomial in y of degree $n(k-1)$. Since $A_k(y)/y$ is a self-reciprocal polynomial, then so is $(A_k(y)/y)^n$, and therefore $a_0, a_1, \dots, a_{nk-n}$ is a palindromic sequence, i.e., $a_j = a_{m-j}$, where for convenience we have set $m := nk - n$. We can then rewrite (4.4) as

$$(4.5) \quad T_{n,k}(z) = \sum_{j=n}^{nk} a_{j-n} \prod_{r=1}^{n(k+1)-1} (nz + j - r) = \sum_{j=0}^m a_j \prod_{r=1}^{n(k+1)-1} (nz + n + j - r),$$

which holds for any integers $n, k \geq 1$. Assuming now that k is even and n is odd, and using symmetry, we get

$$T_{n,k}(z) = \sum_{j=0}^{\lfloor m/2 \rfloor} a_j \left(\prod_{r=1}^{n(k+1)-1} (nz + n + j - r) + \prod_{r=1}^{n(k+1)-1} (nz + nk - j - r) \right),$$

where in the second product j was replaced by $nk - n - j$. We also note that $m = n(k-1)$ is odd since n is odd and k is even; hence the original $m+1$ summands divide evenly into the pairs in the last equation above.

Now we switch the order of multiplication in the second product, replacing r by $nk + n - r$. Then we get

$$(4.6) \quad T_{n,k}(z) = \sum_{j=0}^{\lfloor m/2 \rfloor} a_j \left(\prod_{r=1}^{n(k+1)-1} (nz + n + j - r) + \prod_{r=1}^{n(k+1)-1} (nz - (n + j - r)) \right).$$

For each j , $0 \leq j \leq \lfloor m/2 \rfloor$, there is an r with $1 \leq r \leq n(k+1) - 1$, such that $r = n + j$. Hence the two products in (4.6) are divisible by nz . Next we note that the coefficients of nz in the two products are

$$(4.7) \quad \prod_{\substack{r=1 \\ r \neq n+j}}^{n(k+1)-1} (n+j-r) \quad \text{and} \quad \prod_{\substack{r=1 \\ r \neq n+j}}^{n(k+1)-1} (-(n+j-r)),$$

respectively. Since $n(k+1)$ is odd, then so is the number of factors, $n(k+1) - 2$, in both products (4.7), which are therefore negatives of each other. Hence the coefficient of nz in (4.6) is 0. Replacing n by $n+1$, this proves that $S_{n,k}(z)$ is divisible by z^2 . Finally, divisibility by $(z-1)^2$ now follows from (1.4). \square

By refining the method of proof of Theorem 7 one can, in principle, determine the coefficient of z in $S_{n,k}(z)$, as defined in (1.3), in the case where one of n and k is odd.

Theorem 8. *Let n and k be positive integers, not both even. Then the coefficient of z in $S_{n,k}(z)$ is*

$$(4.8) \quad \frac{(-1)^{k(n+1)-1}(n+1)}{k!((k+1)(n+1)-1)!} \sum_{j=0}^{(k-1)(n+1)} (-1)^j a_j^{(k,n+1)} (n+j)! (k(n+1)-1-j)!,$$

where $a_j^{(k,n+1)}$ is the coefficient of y^j in $(A_k(y)/y)^{n+1}$, with $A_k(y)$ the k th Eulerian polynomial, defined in (2.7).

Proof. By (3.10) and (4.5), the coefficient of z in $S_{n-1,k}(z)$ is the coefficient of z in the expression

$$(4.9) \quad \frac{1}{k!((k+1)n-1)!} \sum_{j=0}^{(k-1)n} a_j^{(k,n)} \prod_{r=1}^{(k+1)n-1} (nz+n+j-r).$$

Now, the coefficient of z in the product in this expression is

$$\begin{aligned} & n \prod_{r=1}^{n+j-1} (n+j-r) \cdot \prod_{r=n+j+1}^{(k+1)n-1} (n+j-r) \\ &= n \prod_{r=1}^{n+j-1} r \cdot \prod_{r=1}^{kn-j-1} (-r) \\ &= (-1)^{kn-j-1} n(n+j-1)!(kn-j-1)!. \end{aligned}$$

Finally we combine this with (4.9), and replace n by $n+1$. This immediately gives the expression (4.8). \square

For the cases $k=1$ and $k=2$ in Theorem 8 we can actually obtain explicit expressions.

Corollary 9. (a) *For any $n \geq 1$, the coefficient of z in $S_{n,1}(z)$ is*

$$\frac{(-1)^n}{\binom{2n+1}{n}}.$$

(b) For any odd $n \geq 1$, the coefficient of z in $S_{n,2}(z)$ is

$$-\frac{(n+1)!^2(n+\frac{n+1}{2})!}{2(3n+2)!(\frac{n+1}{2})!}.$$

Proof. (a) For $k = 1$, the sum in (4.8) consists of a single summand, and since $(A_1(y)/y)^{n+1} = 1$ for all n , the expression (4.8) becomes

$$\frac{(-1)^n(n+1)}{(2n+1)!} \cdot n!^2 = (-1)^n / \binom{2n+1}{n},$$

as claimed.

(b) Since $A_2(y)/y = 1 + y$, we have

$$a_j^{(2,n+1)} = \binom{n+1}{j}, \quad j = 0, 1, \dots, n+1,$$

and the expression (4.8) with $k = 2$ becomes

$$(4.10) \quad \begin{aligned} & \frac{-(n+1)(n+1)!}{2(3n+2)!} \sum_{j=0}^{n+1} (-1)^j \frac{(n+j)!(2n+1-j)!}{j!(n+1-j)!} \\ &= \frac{-(n+1)(n+1)!n!^2}{2(3n+2)!} \sum_{j=0}^{n+1} (-1)^j \binom{n+j}{j} \binom{2n+1-j}{n+1-j}. \end{aligned}$$

By identity (3.36) in [8], this last binomial sum is 0 when n is even (consistent with Theorem 7), and is $\binom{n+(n+1)/2}{(n+1)/2}$ when n is odd. Combining this with (4.10) gives the result of part (b). \square

The coefficient of z in $S_{n,1}(z)$ is obviously the reciprocal of an integer for each $n \geq 1$. Given the form of the corresponding expression in Corollary 9(b), it is rather surprising that this should also hold for the coefficients of z in $S_{n,2}(z)$. In fact, we prove slightly more.

Corollary 10. *For any integer $n \geq 0$, denote*

$$(4.11) \quad c_n := \frac{(n+1)!^2(n+\frac{n+1}{2})!}{2(3n+2)!(\frac{n+1}{2})!},$$

which is defined for all integers $n \geq 0$ if we interpret fractional factorials in terms of the gamma function. Then c_n is the reciprocal of an even integer for all $n \geq 0$.

Before proving this result, we give the first few terms, namely

$$c_0 = \frac{1}{4}, \quad c_1 = \frac{1}{30}, \quad c_2 = \frac{1}{256}, \quad c_3 = \frac{1}{2310}, \quad c_4 = \frac{1}{21504}.$$

The sequence of reciprocals 4, 30, 256, 2310, \dots , is listed as sequence A091527 in [14], with the explicit expansion (in our notation)

$$(4.12) \quad \frac{1}{c_{n-1}} = \frac{(3n)! \Gamma(\frac{n}{2} + 1)}{n!^2 \Gamma(\frac{3n}{2} + 1)}.$$

Proof of Corollary 10. It is easy to verify that (4.12) is consistent with (4.11), and using Euler's duplication formula, c_n can be simplified as

$$c_n = 2^{-2n-2} \frac{\Gamma(\frac{n}{2} + 1) \Gamma(n+2)}{\Gamma(\frac{3n}{2} + 2)} = 2^{-2n-2} \binom{\frac{3n}{2} + 1}{n+1}^{-1}.$$

For even n , the last term contains a binomial coefficient, which gives the statement of the corollary in this case. For odd n , on the other hand, we are dealing with a generalized binomial coefficient which in general is not an integer. We wish to show that for odd n the expression

$$d_n := 2^{2n+2} \binom{\frac{3n}{2} + 1}{n+1}$$

is an integer. We use the fact that binomial coefficients, including generalized ones, satisfy the recurrence relation

$$\binom{\frac{3n}{2} + 1}{n+1} = \binom{\frac{3n}{2}}{n+1} + \binom{\frac{3n}{2}}{n},$$

so the binomial coefficient on the left can be reduced to a linear combination of terms of the form $\binom{1/2}{k}$, with $k \leq n+1$. Hence we are done if we can show that

$$2^{2n+2} \binom{1/2}{k} \in \mathbb{Z}, \quad k = 0, 1, \dots, n+1.$$

By definition, $\binom{1/2}{0} = 1$, and for $k \geq 1$ we have

$$\begin{aligned} \binom{1/2}{k} &= \frac{(\frac{1}{2})(\frac{1}{2}-1) \cdots (\frac{1}{2}-k+1)}{k!} = \frac{(-1)^{k-1}}{2^k k!} \cdot 1 \cdot 3 \cdots (2k-3) \\ &= \frac{(-1)^{k-1}}{2^k k!} \cdot \frac{(2k-2)!}{2^{k-1} k!} = \frac{(-1)^{k-1}}{2^{2k-1}} \cdot \frac{1}{k} \binom{2k-2}{k-1}. \end{aligned}$$

Now $C_{k-1} = \frac{1}{k} \binom{2k-2}{k-1}$ is a Catalan number, which is an integer. With this, we finally have

$$2^{2n+2} \binom{1/2}{k} = (-1)^{k-1} 2^{2(n+1-k)+1} C_{k-1}.$$

This shows that d_n is an even integer when n is odd, and thus for all n . □

Remarks. (1) Corollary 10 can actually be improved as follows. We set

$$a_n := \frac{1}{2(3n+2)c_n} = \frac{(3n+1)!(\frac{n+1}{2})!}{(n+1)!^2(n+\frac{n+1}{2})!} = \frac{2^{2n}}{n+1} \binom{3n/2}{n},$$

where the right-most equality is obtained as in the proof of Corollary 10. The generating function

$$(4.13) \quad A(z) := \sum_{n=0}^{\infty} a_n z^n$$

can then be computed as

$$A(z) = \frac{1}{2z} \left(1 - \cos\left(\frac{1}{3} \arcsin(6\sqrt{3}z)\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{1}{3} \arcsin(6\sqrt{3}z)\right) \right),$$

and with some effort one can show that $A(z)$ satisfies the equation

$$2z^2 A(z)^3 - 3z A(z)^2 + A(z) = 1.$$

Substituting $A(z)$ from (4.13) into this equation, expanding, and then equating coefficients of z^n , we see that each a_n can be written as a sum of products of a_k with $0 \leq k \leq n-1$, with integer coefficients. This implies that all the a_n are

integers, and thus the denominator of c_n is not only even, but is in fact divisible by $2(3n + 2)$. The first few terms of the sequence (a_n) are

$$a_0 = 1, \quad a_1 = 3, \quad a_2 = 16, \quad a_3 = 105, \quad a_4 = 768.$$

This sequence can be found as A085614 in [14]; according to this entry, a_n is the number of elementary arches of size $n + 1$.

(2) Since the coefficients of z in $S_{n,1}(z)$ and $S_{n,2}(z)$ are reciprocals of integers, it would be interesting to know whether this is also the case for $S_{n,3}(z)$. By direct computation using Theorem 8, we find that the first few coefficients of z in $S_{n,3}(z)$ are

$$c_1^{(3)} = -\frac{1}{126}, \quad c_2^{(3)} = -\frac{1}{1155}, \quad c_3^{(3)} = -\frac{1}{6930}, \quad \text{and} \quad c_4^{(3)} = -\frac{10}{513513},$$

which answers the question in the negative. Computing the first 30 terms and using a recurrence fitting algorithm, we found the recurrence relation

$$12(4n + 5)(4n + 11)c_{n+2}^{(3)} - 8(n + 3)(2n + 3)c_{n+1}^{(3)} - (n + 2)(n + 3)c_n^{(3)} = 0.$$

To prove this, we use Theorem 8 with $k = 3$, and write $a_j^{(3,n+1)}$ as a binomial sum. Then we get

$$c_n^{(3)} = \frac{n + 1}{6(4n + 3)!} \sum_{j=0}^{2n+2} (-1)^{n+j} (n + j)! (3n + 2 - j)! \sum_{i=0}^{n+1} \binom{n + 1}{i} \binom{n + 1 - i}{j - 2i} 4^{j-2i}.$$

Finally we apply “creative telescoping” (twice) to this combinatorial sum, for instance by using C. Koutschan’s Mathematica package `HolonomicFunctions` [9].

5. ADDITIONAL REMARKS AND QUESTIONS

1. Using the identity (3.9), which involves a special higher-order Bernoulli polynomial, and using the notation of (1.3), we can rewrite the identity (1.1) as

$$S_{n,0}(z) = \frac{1}{n!} B_n^{(n+1)}((n + 1)z).$$

Similarly, the second part of Theorem 1 can be rephrased to state that for all positive integers n and k , the polynomial $z(z - 1)B_n^{(n+1)}((n + 1)z)$ divides $S_{n,k}(z)$.

2. Related to this, if we divide $S_{n,1}(z)$ by $z(z - 1)B_n^{(n+1)}((n + 1)z)$, we get a sequence of polynomials of degree at most $n - 1$; see Table 1. The first few of these polynomials are listed in Table 3, normalized so that their constant coefficients are 1; we denote them by $p_n(x)$.

n	$p_n(z)$
1	1
2	$1 - 2z$
3	$1 - z + z^2$
4	$1 + \frac{1}{6}z - \frac{13}{2}z^2 + \frac{13}{3}z^3$
5	$1 + \frac{3}{2}z - \frac{27}{2}z^2 + 24z^3 - 12z^4$
6	$1 + \frac{179}{60}z - \frac{473}{24}z^2 + 29z^3 - \frac{571}{24}z^4 + \frac{571}{60}z^5$

Table 3: $p_n(z)$ for $1 \leq n \leq 6$.

It follows from Theorem 1 that these polynomials satisfy the symmetry property $p_n(1-z) = (-1)^{n-1}p_n(z)$, which in turn implies that $p_n(z)$ is divisible by $2z-1$ when n is even. Apart from this, can anything else be said about the polynomials $p_n(z)$? The fact that a relatively large prime (namely 571) appears in the leading coefficient of $p_6(z)$ seems to indicate that the leading coefficients of these polynomials, and indeed of the polynomials $S_{n,k}(z)$, are not as straightforward as the coefficients of z (see, especially, Corollary 9).

3. Again related to the previous point, we note that each product of Bernoulli polynomials in (1.3) has degree

$$\sum_{i=1}^m (k+1+j_i) = m(k+1) + (k+1)(n-m) + k = (k+1)(n+1) - 1,$$

which is independent of m . When k is even, then all terms in (1.3) are positive, and since the Bernoulli polynomials are monic, we may conclude that the degree of $S_{n,k}(z)$ is also $(k+1)(n+1) - 1$. However, this is not clear when k is odd and $S_{n,k}(z)$ is therefore an alternating sum in m .

4. We will now see that there is a close connection between the function $F_k(x, z)$, defined by (2.1), and the *Lerch transcendent* (also known as the *Lerch zeta function*)

$$(5.1) \quad \Phi(z, s, v) := \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s},$$

with $|z| < 1$ and $v \neq 0, -1, -2, \dots$. For various properties and identities see, e.g., [7, Sect. 1.1]. One of these identities is

$$(5.2) \quad \Phi(z, -m, v) = \frac{m!}{z^v} \left(\log \frac{1}{z}\right)^{-m-1} - \frac{1}{z^v} \sum_{r=0}^{\infty} \frac{B_{m+r+1}(v)}{m+r+1} \cdot \frac{(\log z)^r}{r!},$$

where m is a positive integer and $|\log z| < 2\pi$ (see [7, p. 30].) Replacing m, r, v and z by k, m, z and e^x , respectively, (5.2) immediately yields

$$(-x)^{k+1} \frac{e^{xz}}{k!} \Phi(e^x, -k, z) = 1 + (-1)^k \frac{x^{k+1}}{k!} \sum_{m=0}^{\infty} \frac{B_{m+k+1}(z)}{m+k+1} \frac{x^m}{m!},$$

and thus, with (2.1) we have

$$(5.3) \quad F_k(x, z) = (-x)^{k+1} \frac{e^{xz}}{k!} \Phi(e^x, -k, z).$$

Without going into further details, we mention that Boyadzhiev [1, Eq. (6.5)] expresses Lerch transcendents such as the ones on the right of (5.3) in terms of Apostol-Bernoulli polynomials, which in turn can be written as sums involving Eulerian polynomials; see [1, Eq. (4.7)]. Putting everything together, we get (2.10) again, as expected. Furthermore, combining (5.3) with (5.1), we get

$$(5.4) \quad F_k(x, z) = \frac{(-x)^{k+1}}{k!} \sum_{n=0}^{\infty} (n+z)^k e^{(n+z)x},$$

while (2.12) gives

$$F_k(-x, 1-z) = \frac{(-x)^{k+1}}{k!} \sum_{m=1}^{\infty} (m-1+z)^k e^{(1-z-m)x}.$$

The right-hand side of this last identity is the same as that of (5.4), and by (2.2) the two identities are consistent.

5. In addition to Corollary 9, one can say a little more about the coefficients $a_j^{(k,n)}$ that occur in Theorem 8. Indeed, to find an expression for $a_j^{(k,n)} = [y^j](A_k(y)/y)^n$, we apply the multinomial theorem to (2.8), obtaining

$$\begin{aligned} \frac{A_k(y)^n}{y^n} &= \sum_{j_1+j_2+\dots+j_k=n} \binom{n}{j_1, \dots, j_k} A(k, 1)^{j_1} (A(k, 2)y)^{j_2} \dots (A(k, k)y^{k-1})^{j_k} \\ &= \sum_{j_1+j_2+\dots+j_k=n} \binom{n}{j_1, \dots, j_k} A(k, 1)^{j_1} \dots A(k, k)^{j_k} y^{j_2+2j_3+\dots+(k-1)j_k}, \end{aligned}$$

and therefore

$$(5.5) \quad \left(\frac{A_k(y)}{y} \right)^n = \sum_{j=0}^{n(k-1)} \left(\sum_{j_1, \dots, j_k} \binom{n}{j_1, \dots, j_k} A(k, 1)^{j_1} \dots A(k, k)^{j_k} \right) y^j,$$

where the inner sum is taken over all j_1, \dots, j_k satisfying

$$(5.6) \quad \begin{cases} j_1 + j_2 + \dots + j_k = n, \\ j_2 + 2j_3 + \dots + (k-1)j_k = j, \end{cases}$$

Now, by definition and (5.6) we have

$$a_j^{(k,n)} = \sum_{j_1, \dots, j_k} \binom{n}{j_1, \dots, j_k} A(k, 1)^{j_1} A(k, 2)^{j_2} \dots A(k, k)^{j_k},$$

where the summation is again over all j_1, \dots, j_k satisfying (5.6). This identity can be rewritten as

$$(5.7) \quad a_j^{(k,n)} = \sum_{j_1=0}^n \frac{n!}{j_1!(n-j_1)!} \sum_{j_2, \dots, j_k} \binom{n-j_1}{j_2, \dots, j_k} A(k, 2)^{j_2} \dots A(k, k)^{j_k},$$

where we have used the fact that $A(k, 1) = 1$, and the inner sum is over all j_2, \dots, j_k satisfying

$$\begin{cases} j_2 + \dots + j_k = n - j_1, \\ j_2 + 2j_3 + \dots + (k-1)j_k = j. \end{cases}$$

Finally, setting $r := n - j_1$, we can slightly simplify (5.7) as

$$(5.8) \quad a_j^{(k,n)} = \sum_{r=0}^n \binom{n}{r} \sum_{j_2, \dots, j_k} \binom{r}{j_2, \dots, j_k} A(k, 2)^{j_2} \dots A(k, k)^{j_k},$$

where the inner sum is over all j_2, \dots, j_k satisfying

$$\begin{cases} j_2 + \dots + j_k = r, \\ j_2 + 2j_3 + \dots + (k-1)j_k = j. \end{cases}$$

This last inner sum is reminiscent of an (ordinary) Bell polynomial, but is still somewhat different.

6. For a different approach to the coefficients $a_j^{(k,n)}$ we use the infinite series (2.9) and rewrite it as

$$(5.9) \quad \frac{\frac{1}{y} A_k(y)}{(1-y)^{k+1}} = \sum_{j=0}^{\infty} (j+1)^k y^j,$$

for $k \geq 1$. Raising both sides of (5.9) to the n th power, we get

$$(5.10) \quad \frac{(A_k(y)/y)^n}{(1-y)^{(k+1)n}} = \sum_{\nu=0}^{\infty} \left(\sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 + \dots + j_n = \nu}} ((j_1 + 1) \cdots (j_n + 1))^k \right) y^\nu.$$

We denote the inner multiple sum by $u_\nu^{(k,n)}$ and rewrite it as

$$(5.11) \quad u_\nu^{(k,n)} = \sum_{\substack{j_1, \dots, j_n \geq 1 \\ j_1 + \dots + j_n = \nu + n}} (j_1 \cdots j_n)^k.$$

Using (5.10), (5.11) and the definition of $a_j^{(k,n)}$, we get

$$a_j^{(k,n)} = \sum_{\nu=0}^j (-1)^{j-\nu} \binom{(k+1)n}{j-\nu} u_\nu^{(k,n)}.$$

It is therefore of interest to find out more about the numbers $u_\nu^{(k,n)}$.

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