Ramanujan's τ Function

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Outline

The τ Function (I)

Definition

$$\Delta \triangleq q \prod_{n \geqslant 1} (1 - q^n)^{24} = \sum_{n \geqslant 1} \tau(n) q^n.$$

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Example

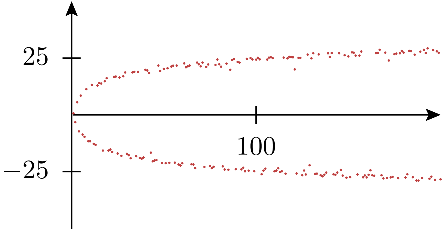
The first values are

$$\begin{aligned} \tau(1) &= 1 & \tau(5) &= 4830 \\ \tau(2) &= -24 & \tau(6) &= -6048 \\ \tau(3) &= 252 & \tau(7) &= -16744 \\ \tau(4) &= -1472 & \tau(8) &= 84480 \end{aligned}$$

...

The τ Function (II)

- A plot of $log(\tau(n))$.



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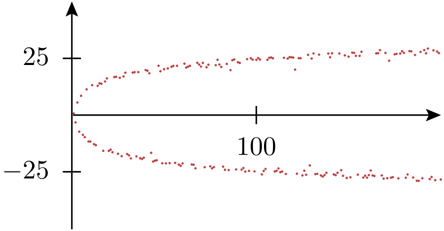

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$$|\tau(p)| \leqslant 2p^{11/2}.$$

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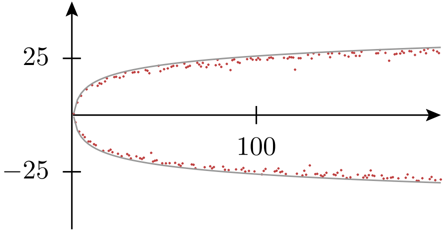

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Simple Congruences (I)

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Theorem

$$\begin{aligned} \tau(mn) &= \tau(m)\tau(n) \quad \textit{if } \gcd(m,n)=1, \\ \tau(p^{n+1}) &= \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}) \quad \textit{if } p \textit{ prime.} \end{aligned}$$

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- Only a few such primes are known:

$$2,\ 3,\ 5,\ 7,\ 2411$$

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. . .

Eisenstein Series

- Eisenstein Series E_n are an example of modular forms of weight $2n$.

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- Any modular form f can be obtained as a polynomial in E_2 and E_3.
- What about E_1?
- E_1 is not modular but

$$E_1(-1/z) = z^2 E_1(z) + \frac{12}{2\pi i} z.$$

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Lemma

If f is a modular form of weight k then

$$\theta f - \frac{k}{12} E_1 f$$

is a modular form of weight $k + 2$.

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- And the nice congruences

$$n \equiv 0, 2 \pmod 6 \quad \Longrightarrow \quad \tau(n) \equiv 0 \pmod{24}.$$

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An Exact Formula

- $\Delta = \alpha E_3^2 + \beta E_6$.

$$E_3 = 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n$$

$$E_6 = 1 + \frac{65520}{691} \sum_{n \geqslant 1} \sigma_{11}(n) q^n$$

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- $762048 \Delta = -691 E_3^2 + 691 E_6$.
- In other words,

$$\tau(n) = \frac{691}{756} \sigma_5(n) - \frac{691}{3} \sigma_5 * \sigma_5(n) + \frac{65}{756} \sigma_{11}(n).$$

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Modulus 691

- Consider the previous

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$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

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- For primes

$$\tau(p) \equiv 1 + p^{11} \pmod{691}.$$

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Modulus 7

- Let's start with

$$\begin{aligned} E_3 &\equiv 1, \\ E_2^2 = E_4 &\equiv E_1 \pmod{7}. \end{aligned}$$

$$E_2 = 1 + 240 \sum_{n \geqslant 1} \sigma_3(n) q^n$$

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- So that

$$\tau(n) \equiv n\sigma_3(n) \pmod{7}.$$

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Ramanujan Can Err

- We have

$$\begin{aligned} \tau(n) &\equiv n\sigma_1(n) \pmod{5} \\ \tau(n) &\equiv n\sigma_9(n) \pmod{25} \\ \tau(n) &\equiv n^{41}\sigma_{29}(n) \pmod{125}. \end{aligned}$$

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- This is false for $k \geqslant 4$.
- Take $n = 443$. Since $443^2 \equiv -1 \pmod{5^4}$.

$$\tau(443) = 328369848718692 \equiv 567 \not\equiv 443^a(1+443^b) \pmod{5^4}$$

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- Which implies

$$\tau(p) = 0 \implies \left\{ \begin{array}{l} p \equiv -1 \pmod{5^2 \cdot 691} \\ p \equiv -1, 3, 5 \pmod{7} \end{array} \right\}$$
$$\implies p = 863749, 1381999, 1589299, \ldots.$$

...

Open Problems (II)

- Is $\tau(n)$ ever a prime? Indeed,

$$\begin{aligned} \tau(63001) &= \tau(251^2) = \tau(251)^2 - 251^{11} \\ &= -80561663527802406257321747. \end{aligned}$$

Open Problems (II)

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- More of the following sort?

$$\begin{aligned}\tau(p) \equiv 0 \pmod{p} &\implies p = 2, 3, 5, 7, 2411, \ldots \\ \tau(p) \equiv 1 \pmod{p} &\implies p = 11, 23, 691, \ldots \\ \tau(p) \equiv -1 \pmod{p} &\implies p = 5807, \ldots\end{aligned}$$