Ramanujan's τ Function

Armin Straub

06-May 2007

Outline

The τ Function (I)

Definition

$$\Delta \triangleq q \prod_{n \geqslant 1} (1 - q^n)^{24} = \sum_{n \geqslant 1} \tau(n) q^n.$$

The τ Function (I)

Definition

$$\Delta \triangleq q \prod_{n \geqslant 1} (1 - q^n)^{24} = \sum_{n \geqslant 1} \tau(n) q^n.$$

Example

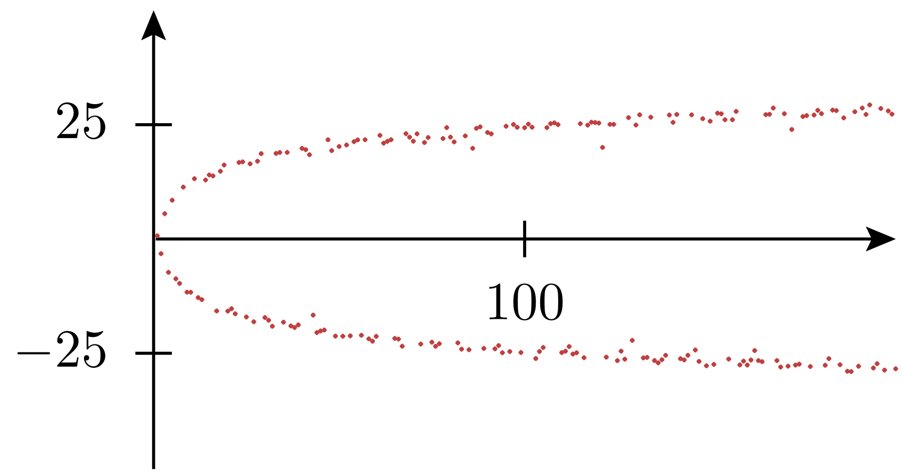
The first values are

$$\begin{aligned} \tau(1) &= 1 & \tau(5) &= 4830 \\ \tau(2) &= -24 & \tau(6) &= -6048 \\ \tau(3) &= 252 & \tau(7) &= -16744 \\ \tau(4) &= -1472 & \tau(8) &= 84480 \end{aligned}$$

...

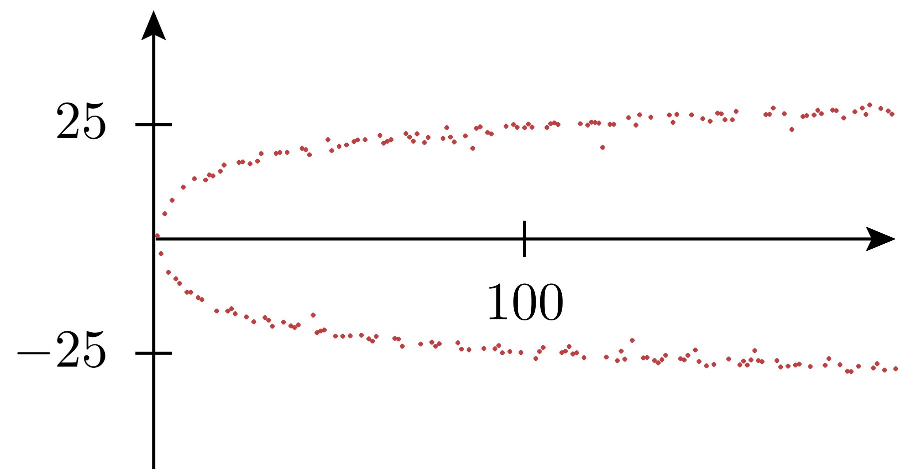
The τ Function (II)

- A plot of $log(\tau(n))$.



The τ Function (II)

- A plot of $log(\tau(n))$.

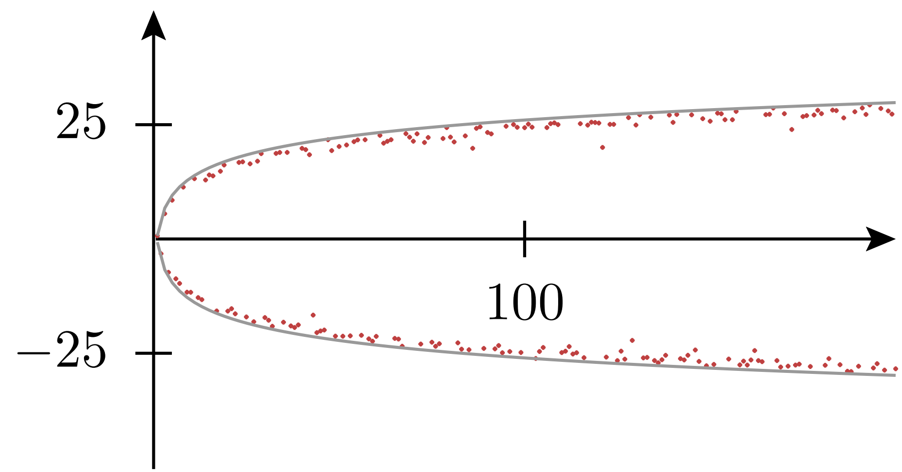


- Ramanujan conjectured and Deligne proved

$$|\tau(p)| \leqslant 2p^{11/2}.$$

The τ Function (II)

- A plot of $log(\tau(n))$.



- Ramanujan conjectured and Deligne proved

$$|\tau(p)| \leqslant 2p^{11/2}.$$

...

Simple Congruences (I)

- $(a+b)^p \equiv a^p + b^p \pmod{p}$

Simple Congruences (I)

- $(a+b)^p \equiv a^p + b^p \pmod{p}$
- Hence,

$$q\prod_{n\geqslant 1}(1-q^n)^{24} \equiv q\prod_{n\geqslant 1}(1-q^n)^3\prod_{n\geqslant 1}(1-q^{7n})^3 \pmod{7}$$

Simple Congruences (I)

- $(a+b)^p \equiv a^p + b^p \pmod{p}$
- Hence, using Jacobi's identity

$$\begin{aligned} q\prod_{n\geqslant 1}(1-q^n)^{24} &\equiv q\prod_{n\geqslant 1}(1-q^n)^3\prod_{n\geqslant 1}(1-q^{7n})^3 \pmod{7} \\ &= q\sum_{n\geqslant 0}(-1)^n(2n+1)q^{n(n+1)/2}\prod_{n\geqslant 1}(1-q^{7n})^3 \end{aligned}$$

Simple Congruences (I)

- $(a+b)^p \equiv a^p + b^p \pmod{p}$
- Hence, using Jacobi's identity

$$\begin{aligned} q\prod_{n\geqslant 1}(1-q^n)^{24} &\equiv q\prod_{n\geqslant 1}(1-q^n)^3\prod_{n\geqslant 1}(1-q^{7n})^3 \pmod{7} \\ &= q\sum_{n\geqslant 0}(-1)^n(2n+1)q^{n(n+1)/2}\prod_{n\geqslant 1}(1-q^{7n})^3 \end{aligned}$$

- $\frac{n(n+1)}{2} + 1 \equiv 0, 1, 2, 4 \pmod{7}$

Simple Congruences (I)

- $(a+b)^p \equiv a^p + b^p \pmod{p}$
- Hence, using Jacobi's identity

$$q \prod_{n \geqslant 1} (1-q^n)^{24} \equiv q \prod_{n \geqslant 1} (1-q^n)^3 \prod_{n \geqslant 1} (1-q^{7n})^3 \pmod{7}$$
$$= q \sum_{n \geqslant 0} (-1)^n (2n+1) q^{n(n+1)/2} \prod_{n \geqslant 1} (1-q^{7n})^3$$

- $\frac{n(n+1)}{2} + 1 \equiv 0, 1, 2, 4 \pmod{7}$
- We conclude

$$\tau(7n+3), \tau(7n+5), \tau(7n+6) \equiv 0 \pmod{7}.$$

Simple Congruences (I)

- $(a+b)^p \equiv a^p + b^p \pmod{p}$
- Hence, using Jacobi's identity

$$\begin{aligned} q\prod_{n\geqslant 1}(1-q^n)^{24} &\equiv q\prod_{n\geqslant 1}(1-q^n)^3\prod_{n\geqslant 1}(1-q^{7n})^3 \pmod{7} \\ &= q\sum_{n\geqslant 0}(-1)^n(2n+1)q^{n(n+1)/2}\prod_{n\geqslant 1}(1-q^{7n})^3 \end{aligned}$$

- $\frac{n(n+1)}{2}+1 \equiv 0,1,2,4 \pmod{7}$
- We conclude

$$\tau(7n),\tau(7n+3),\tau(7n+5),\tau(7n+6) \equiv 0 \pmod{7}.$$

...

Simple Congruences (II)

Theorem

$$\begin{aligned} \tau(mn) &= \tau(m)\tau(n) \quad \textit{if } \gcd(m,n)=1, \\ \tau(p^{n+1}) &= \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}) \quad \textit{if } p \textit{ prime.} \end{aligned}$$

Simple Congruences (II)

Theorem

$$\begin{aligned} \tau(mn) &= \tau(m)\tau(n) \quad \textit{if } \gcd(m,n)=1, \\ \tau(p^{n+1}) &= \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}) \quad \textit{if } p \textit{ prime}. \end{aligned}$$

- $\tau(p) \equiv 0 \pmod{p} \implies \tau(np) \equiv 0 \pmod{p}$

Simple Congruences (II)

Theorem

$$\begin{aligned} \tau(mn) &= \tau(m)\tau(n) && \textit{if } \gcd(m,n)=1, \\ \tau(p^{n+1}) &= \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}) && \textit{if } p \textit{ prime.} \end{aligned}$$

- $\tau(p) \equiv 0 \pmod{p} \quad \Longrightarrow \quad \tau(np) \equiv 0 \pmod{p}$
- $\tau(7) = -16744 \quad \Longrightarrow \quad \tau(7n) \equiv 0 \pmod{7}$

Simple Congruences (II)

Theorem

$$\begin{aligned} \tau(mn) &= \tau(m)\tau(n) \quad \textit{if } \gcd(m,n)=1, \\ \tau(p^{n+1}) &= \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}) \quad \textit{if } p \textit{ prime.} \end{aligned}$$

- $\tau(p) \equiv 0 \pmod{p} \implies \tau(np) \equiv 0 \pmod{p}$
- $\tau(7) = -16744 \implies \tau(7n) \equiv 0 \pmod{7}$
- Only a few such primes are known:

$$2,\ 3,\ 5,\ 7,\ 2411$$

.

...

Eisenstein Series

- Eisenstein Series E_n are an example of modular forms of weight $2n$.

$$E_n = 1 - \frac{4k}{B_{2k}} \sum_{n \geqslant 1} \sigma_{2k-1}(n) q^n$$

Eisenstein Series

- Eisenstein Series E_n are an example of modular forms of weight $2n$.

$$E_n = 1 - \frac{4k}{B_{2k}} \sum_{n \geqslant 1} \sigma_{2k-1}(n) q^n$$

- Any modular form f can be obtained as a polynomial in E_2 and E_3.

Eisenstein Series

- Eisenstein Series E_n are an example of modular forms of weight $2n$.

$$E_n = 1 - \frac{4k}{B_{2k}} \sum_{n \geqslant 1} \sigma_{2k-1}(n) q^n$$

- Any modular form f can be obtained as a polynomial in E_2 and E_3.
- What about E_1?

Eisenstein Series

- Eisenstein Series E_n are an example of modular forms of weight $2n$.

$$E_n = 1 - \frac{4k}{B_{2k}} \sum_{n \geqslant 1} \sigma_{2k-1}(n) q^n$$

- Any modular form f can be obtained as a polynomial in E_2 and E_3.
- What about E_1?
- E_1 is not modular but

$$E_1(-1/z) = z^2 E_1(z) + \frac{12}{2\pi i} z.$$

...

Differentiating Modular Forms

- If f is a modular form. What about

$$\frac{\mathrm{d}f}{\mathrm{d}z}\ ?$$

Differentiating Modular Forms

- If f is a modular form. What about

$$q\frac{\mathrm{d}f}{\mathrm{d}q} = \frac{1}{2\pi i}\frac{\mathrm{d}f}{\mathrm{d}z} \ ?$$

Differentiating Modular Forms

- If f is a modular form. What about

$$\theta f \triangleq q\frac{\mathrm{d}f}{\mathrm{d}q} = \frac{1}{2\pi i}\frac{\mathrm{d}f}{\mathrm{d}z}\ ?$$

Differentiating Modular Forms

- If f is a modular form. What about

$$\theta f \triangleq q \frac{\mathrm{d}f}{\mathrm{d}q} = \frac{1}{2\pi i} \frac{\mathrm{d}f}{\mathrm{d}z} \ ?$$

- Not modular, but this is where E_1 comes in.

Differentiating Modular Forms

- If f is a modular form. What about

$$\theta f \triangleq q \frac{\mathrm{d} f}{\mathrm{d} q} = \frac{1}{2\pi i} \frac{\mathrm{d} f}{\mathrm{d} z} \ ?$$

- Not modular, but this is where E_1 comes in.

Lemma

If f is a modular form of weight k then

$$\theta f - \frac{k}{12} E_1 f$$

is a modular form of weight $k + 2$.

...

Differentiating Δ

$$\theta f - \frac{k}{12} E_1 f$$

- Applied to Δ which is of weight 12,

$$\theta\Delta - E_1\Delta = 0.$$

Differentiating Δ

$$E_1 = 1 - 24 \sum_{n \geqslant 1} \sigma_1(n) q^n$$

- Applied to Δ which is of weight 12,

$$\theta\Delta - E_1\Delta = 0.$$

- This gives the recursion

$$(n-1)\tau(n) = -24 \sum_{m=1}^{n-1} \tau(m)\sigma_1(n-m).$$

Differentiating Δ

$$E_1 = 1 - 24 \sum_{n \geqslant 1} \sigma_1(n) q^n$$

- Applied to Δ which is of weight 12,

$$\theta\Delta - E_1\Delta = 0.$$

- This gives the recursion

$$(n-1)\tau(n) = -24 \sum_{m=1}^{n-1} \tau(m)\sigma_1(n-m).$$

- And the nice congruences

$$n \equiv 0, 2 \pmod{6} \quad \Longrightarrow \quad \tau(n) \equiv 0 \pmod{24}.$$

. . .

An Exact Formula

- $\Delta = \alpha E_3^2 + \beta E_6.$

$$\begin{aligned} E_3 &= 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n \\ E_6 &= 1 + \frac{65520}{691} \sum_{n \geqslant 1} \sigma_{11}(n) q^n \end{aligned}$$

An Exact Formula

$$E_3 = 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n$$

$$E_6 = 1 + \frac{65520}{691} \sum_{n \geqslant 1} \sigma_{11}(n) q^n$$

- $\Delta = \alpha E_3^2 + \beta E_6$.
- This requires

$$\begin{aligned} 0 &= \alpha + \beta, \\ 1 &= -2 \cdot 504\alpha + \frac{65520}{691}\beta. \end{aligned}$$

An Exact Formula

$$E_3 = 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n$$
$$E_6 = 1 + \frac{65520}{691} \sum_{n \geqslant 1} \sigma_{11}(n) q^n$$

- $\Delta = \alpha E_3^2 + \beta E_6$.
- This requires

$$\begin{aligned} 0 &= \alpha + \beta, \\ 1 &= -2 \cdot 504\alpha + \frac{65520}{691}\beta. \end{aligned}$$

- $762048\Delta = -691E_3^2 + 691E_6$.

An Exact Formula

$$E_3 = 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n$$
$$E_6 = 1 + \frac{65520}{691} \sum_{n \geqslant 1} \sigma_{11}(n) q^n$$

- $\Delta = \alpha E_3^2 + \beta E_6$.
- This requires

$$\begin{aligned} 0 &= \alpha + \beta, \\ 1 &= -2 \cdot 504\alpha + \frac{65520}{691}\beta. \end{aligned}$$

- $762048\Delta = -691 E_3^2 + 691 E_6$.
- In other words,

$$\tau(n) = \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sigma_5 * \sigma_5(n) + \frac{65}{756}\sigma_{11}(n).$$

...

Modulus 691

- Consider the previous

$$\tau(n) = \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sigma_5 * \sigma_5(n) + \frac{65}{756}\sigma_{11}(n).$$

Modulus 691

- Consider the previous

$$\tau(n) = \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sigma_5 * \sigma_5(n) + \frac{65}{756}\sigma_{11}(n).$$

- As desired,

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

Modulus 691

- Consider the previous

$$\tau(n) = \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sigma_5 * \sigma_5(n) + \frac{65}{756}\sigma_{11}(n).$$

- As desired,

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

- For primes

$$\tau(p) \equiv 1 + p^{11} \pmod{691}.$$

...

Modulus 7

- Let's start with

$$\begin{aligned} E_3 &\equiv 1, \\ E_2^2 = E_4 &\equiv E_1 \pmod{7}. \end{aligned}$$

$$\begin{aligned} E_2 &= 1 + 240 \sum_{n \geqslant 1} \sigma_3(n) q^n \\ E_3 &= 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n \\ E_4 &= 1 + 480 \sum_{n \geqslant 1} \sigma_7(n) q^n \end{aligned}$$

$$\theta f - \frac{k}{12} E_1 f$$

Modulus 7

- Let's start with

$$\begin{aligned} E_3 &\equiv 1, \\ E_2^2 = E_4 &\equiv E_1 \pmod{7}. \end{aligned}$$

- Then

$$\begin{aligned} 1728\Delta &= E_2^3 - E_3^2 \\ &\equiv E_1 E_2 - E_3 \\ &= 3\theta E_2 \pmod{7}. \end{aligned}$$

$$\begin{aligned} E_2 &= 1 + 240 \sum_{n \geqslant 1} \sigma_3(n) q^n \\ E_3 &= 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n \\ E_4 &= 1 + 480 \sum_{n \geqslant 1} \sigma_7(n) q^n \end{aligned}$$

$$\theta f - \frac{k}{12} E_1 f$$

Modulus 7

- Let's start with

$$\begin{aligned} E_3 &\equiv 1, \\ E_2^2 = E_4 &\equiv E_1 \pmod 7. \end{aligned}$$

- Then

$$\begin{aligned} 1728\Delta &= E_2^3 - E_3^2 \\ &\equiv E_1 E_2 - E_3 \\ &= 3\theta E_2 \pmod 7. \end{aligned}$$

- So that

$$\tau(n) \equiv n\sigma_3(n) \pmod 7.$$

$$\begin{aligned} E_2 &= 1 + 240 \sum_{n \geqslant 1} \sigma_3(n) q^n \\ E_3 &= 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n \\ E_4 &= 1 + 480 \sum_{n \geqslant 1} \sigma_7(n) q^n \end{aligned}$$

$$\theta f - \frac{k}{12} E_1 f$$

...

Ramanujan Can Err

- We have

$$\begin{aligned}\tau(n) &\equiv n\sigma_1(n) \pmod{5}\\ \tau(n) &\equiv n\sigma_9(n) \pmod{25}\\ \tau(n) &\equiv n^{41}\sigma_{29}(n) \pmod{125}.\end{aligned}$$

Ramanujan Can Err

- We have

$$\begin{aligned} \tau(n) &\equiv n\sigma_1(n) \pmod{5} \\ \tau(n) &\equiv n\sigma_9(n) \pmod{25} \\ \tau(n) &\equiv n^{41}\sigma_{29}(n) \pmod{125}. \end{aligned}$$

- Ramanujan conjectured that

$$\tau(n) \equiv n^a \sigma_b(n) \pmod{5^k}.$$

Ramanujan Can Err

- We have

$$\begin{aligned}\tau(n) &\equiv n\sigma_1(n) \pmod{5}\\ \tau(n) &\equiv n\sigma_9(n) \pmod{25}\\ \tau(n) &\equiv n^{41}\sigma_{29}(n) \pmod{125}.\end{aligned}$$

- Ramanujan conjectured that

$$\tau(n) \equiv n^a\sigma_b(n) \pmod{5^k}.$$

- This is false for $k \geqslant 4$.

Ramanujan Can Err

- We have

$$\begin{aligned} \tau(n) &\equiv n\sigma_1(n) \pmod{5} \\ \tau(n) &\equiv n\sigma_9(n) \pmod{25} \\ \tau(n) &\equiv n^{41}\sigma_{29}(n) \pmod{125}. \end{aligned}$$

- Ramanujan conjectured that

$$\tau(n) \equiv n^a \sigma_b(n) \pmod{5^k}.$$

- This is false for $k \geqslant 4$.
- Take $n = 443$. Since $443^2 \equiv -1 \pmod{5^4}$.

$$\tau(443) = 328369848718692 \equiv 567 \not\equiv 443^a(1+443^b) \pmod{5^4}$$

…

Open Problems (I)

- Of course, Lehmer's conjecture

$$\tau(n) \neq 0.$$

Open Problems (I)

- Of course, Lehmer's conjecture

$$\tau(n) \neq 0.$$

- We found

$$\begin{aligned} \tau(p) &\equiv p(1+p^9) \pmod{25} \\ \tau(p) &\equiv p(1+p^3) \pmod{7} \\ \tau(p) &\equiv 1+p^{11} \pmod{691}. \end{aligned}$$

Open Problems (I)

- Of course, Lehmer's conjecture

$$\tau(n) \neq 0.$$

- We found

$$\begin{aligned}
\tau(p) &\equiv p(1+p^9) \pmod{25} \\
\tau(p) &\equiv p(1+p^3) \pmod{7} \\
\tau(p) &\equiv 1+p^{11} \pmod{691}.
\end{aligned}$$

- Which implies

$$\tau(p) = 0 \implies \left\{ \begin{array}{l} p \equiv -1 \pmod{5^2 \cdot 691} \\ p \equiv -1, 3, 5 \pmod{7} \end{array} \right\}$$
$$\implies p = 863749, 1381999, 1589299, \ldots .$$

...

Open Problems (II)

- Is $\tau(n)$ ever a prime? Indeed,

$$\begin{aligned}\tau(63001) &= \tau(251^2) = \tau(251)^2 - 251^{11} \\ &= -80561663527802406257321747.\end{aligned}$$

Open Problems (II)

- Is $\tau(n)$ ever a prime? Indeed,

$$\begin{aligned}\tau(63001) &= \tau(251^2) = \tau(251)^2 - 251^{11} \\ &= -80561663527802406257321747.\end{aligned}$$

- More of the following sort?

$$\begin{aligned}\tau(p) \equiv 0 \pmod{p} &\implies p = 2, 3, 5, 7, 2411, \ldots \\ \tau(p) \equiv 1 \pmod{p} &\implies p = 11, 23, 691, \ldots \\ \tau(p) \equiv -1 \pmod{p} &\implies p = 5807, \ldots\end{aligned}$$