Problems #12

Problem 1. (3 XP extra) Consider an identity of the form

$$\sum_k f(n,k) = A(n), \tag{1}$$

where $f(n,k)$ and $A(n)$ are hypergeometric terms in n and k.

It is a striking empirical observation that $F(n,k) = f(n,k)/A(n)$ frequently[1] has a WZ mate $G(n,k)$. This is often referred to as the WZ miracle. Find an identity of the form (1) for which this miracle does not occur.

Problem 2. (2 XP) Using Zeilberger's algorithm, discover and prove the following identities (that is, proceed without knowledge of the right-hand sides).

(a) The Gauss summation formula

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

in the special case $a = -n$, with $n \in \mathbb{Z}_{\geqslant 0}$ (also known as the Chu-Vandermonde identity).

(b) For $n \in \mathbb{Z}_{\geqslant 0}$, Saalschütz's theorem

$$ {}_3F_2\left(\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix} \middle| 1\right) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}.$$

Problem 3. (3 XP) Show in many different ways that

$$\sum_{k=1}^{n} k \binom{n}{k} = n 2^{n-1}.$$

(a) Combinatorially.

(b) Using exponential generating functions.

(c) Using ordinary generating functions.

(d) Using Sister Celine's approach.

(e) Using a WZ pair.

(f) Using Zeilberger's algorithm.

Bibliography

[PWZ96] Marko Petkovsek, Herbert S. Wilf, and Doron Zeilberger. *A=B*. A. K. Peters, 1996.
[Tef04] Akalu Tefera. What is... a Wilf-Zeilberger pair. *AMS Notices*, 57, 2004.
[WZ90] Herbert S. Wilf and Doron Zeilberger. Rational functions certify combinatorial identities. *Journal of the American Mathematical Society*, 3(1):147–158, 1990.

1. "very often" according to [WZ90], 99% of the time according to [PWZ96, p. 123], and 99.99% of the time according to [Tef04]

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