Problems #11

Problem 1. (1 XP) Determine whether the following are hypergeometric terms in both n and k. Are they proper?

$$a_{n,k} = \frac{(-1)^n}{n}\binom{2n}{n+k}, \quad b_{n,k} = \frac{1}{n+k+1}, \quad c_{n,k} = \frac{1}{n^2+k^2+1}$$

Problem 2. (2 XP) Approach the following problems using Celine's method (see our course website for instructions how to obtain an implementation for Sage).

(a) Evaluate the sum $\sum_{k \geqslant 0} \binom{n-k}{k}$.

(b) Evaluate the sums $\sum_k \binom{n}{k}^a$ for $a=1$ and $a=2$. Find a recursion for the case $a=3$. What about $a=4$?

(c) Find a recursion for the Apéry numbers

$$A(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{k},$$

which can be used to prove the irrationality of $\zeta(2)$.

(d) Determine the three-term recursion for the Laguerre polynomials $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}$.

Problem 3. (2 XP) Consider the two operators $A = S_n - (n+1)$ and $B = nS_n - 2(n+1)$.

(a) Determine solutions a_n and b_n to the equations $Aa_n = 0$ and $Bb_n = 0$.

(b) Compute the product AB.

(c) Verify that we have $ABb_n = 0$ but $ABa_n \neq 0$. Why is that to be expected?

(d) Find an operator C such that $Ca_n = 0$ and $Cb_n = 0$.

Hint: The least common left multiple of A and B is the minimal operator L such that $L = UA$ and $L = VB$ for some operators U, V.

Problem 4. (2 XP)

(a) We want to show that a sequence A_n is zero. Suppose we know that, for all $n \geqslant 0$,

$$[p_2(n)S_n^2 + p_1(n)S_n + p_0(n)]A_n = 0,$$

with polynomials coefficients $p_i(n)$. We check that $A_0 = 0, A_1 = 0, \ldots$ are indeed zero. After how many initial values are we able to conclude that $A_n = 0$ for all $n \geqslant 0$? *Hint:* Two may not be enough!

(b) Prove Strehl's identity

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}.$$

Hint: A least common left multiple as in the previous problem might be handy.

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