Problems #10

Problem 1. (2 XP) Express the following functions in terms of hypergeometric functions.

(a) $\sin(x)$

(b) $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t$

(c) $\log(1+x)$

Problem 2. (2 XP) Glaisher recorded in 1874 the integral evaluation

$$\int_0^\infty (a_0 - a_1 x^2 + a_2 x^4 - \cdots) \mathrm{d}x = \frac{\pi}{2} a_{-1/2},$$

and comments that, in the examples he worked out, the term $a_{-1/2}$ can be made sense of when a_n is explicitly given by factorial ratios.

(a) Verify the simple special case $a_n = 1/n!$.

(b) Can you (formally) derive this evaluation by rewriting the integrand in terms of the shift operator S?

(c) Show that Glaisher's formula is a special case of Ramanujan's master theorem.

Problem 3. (2 XP) A solution of the Bessel differential equation $x^2 y'' + x y' + (x^2 - \alpha^2) y = 0$ is the Bessel function

$$J_\alpha(x) = \sum_{n \geqslant 0} \frac{(-1)^n}{n! \Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{2n+\alpha}$$

of the first kind.

(a) Express $J_\alpha(x)$ in terms of a hypergeometric function.

(b) Determine the Mellin transform of the Bessel function $J_\alpha(x)$, that is, evaluate, for $\operatorname{Re}(s + \alpha) > 0$,

$$\int_0^\infty x^{s-1} J_\alpha(x) \, \mathrm{d}x.$$

(You may assume that the conditions, which we didn't discuss yet, of Ramanujan's master theorem are satisfied.)

Problem 4. (2 XP)

(a) Apply Celine's algorithm by hand to $a_{n,k} = \binom{n}{k}$ to find a recurrence $P(n, S_n, S_k) \binom{n}{k} = 0$.

(b) Conclude that, as we all know, $\sum_k \binom{n}{k} = 2^n$.

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