Problems #7

Problem 1. (2 XP) Let $\mathcal{X}$ be the vector space of solutions to the differential equation

$$y^{(d)} + c_{d-1} y^{(d-1)} + \ldots + c_1 y' + c_0 y = 0,$$

and let $\mathcal{Y}$ be the vector space of solutions to the recurrence

$$a_{n+d} + c_{d-1} a_{n+d-1} + \ldots + c_1 a_{n+1} + c_0 a_n = 0.$$

Show that the map EGF: $\mathcal{Y} \to \mathcal{X}$ defined by $(a_n)_{n \geqslant 0} \mapsto \sum_{n \geqslant 0} a_n \frac{x^n}{n!}$ is an isomorphism.

Solution. The map is clearly linear and injective.

Since $\mathcal{X}$ and $\mathcal{Y}$ both have dimension d (the number of initial conditions needed to describe a unique solution), it only remains to show that the map is actually well-defined, that is, that it sends solutions of the recurrence to solutions of the differential equation.

This, however, is a direct consequence of the fact that

$$\mathrm{EGF}(S a_n) = D\,\mathrm{EGF}(a_n),$$

which was already observed in an earlier problem (if $f(x)$ is the egf of the sequence a_n, then $f'(x)$ is the egf of a_{n+1}). ☐

Problem 2. (1 XP) True or false? Any eventually periodic sequence is C-finite.

Solution. True. Suppose the sequence $(a_n)_{n \geqslant 0}$ is eventually periodic with period T, that is, $a_{n+T} = a_n$ for all $n \geqslant N$. Equivalently, $a_{n+T+N} = a_{n+N}$ for all $n \geqslant 0$. This is a linear recurrence with constant coefficients (the corresponding operator is $(S^T - 1)S^N$), and so the sequence is C-finite. ☐

Problem 3. (2 XP) The Chebyshev polynomials $T_n(x)$ of the first kind are the unique polynomials satisfying

$$T_n(\cos\theta) = \cos(n\theta).$$

Prove that the sequence $(T_n(x))_{n \geqslant 0}$ is C-finite.

Solution. Write $x = \cos\theta$. Then,

$$\begin{aligned} \sum_{n \geqslant 0} T_n(x) z^n &= \sum_{n \geqslant 0} \cos(n\theta) z^n = \frac{1}{2} \sum_{n \geqslant 0} (e^{in\theta} + e^{-in\theta}) z^n \\ &= \frac{1}{2}\left(\frac{1}{1 - e^{i\theta} z} + \frac{1}{1 - e^{-i\theta} z}\right) = \frac{1}{2} \frac{2 - 2(\cos\theta) z}{1 - 2(\cos\theta) z + z^2} = \frac{1 - xz}{1 - 2xz + z^2}, \end{aligned}$$

so that, in particular, the sequence $(T_n(x))_{n \geqslant 0}$ is C-finite.

Alternatively, we can use the trigonometric identity

$$\cos(n\theta) = 2\cos(\theta)\cos((n-1)\theta) - \cos((n-2)\theta)$$

to derive the recurrence

$$T_{n+2}(x) = 2x T_{n+1}(x) - T_n(x),$$

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which again shows the C-finiteness.

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Sage] [chebyshev_T(n,x) for n in [0..5]]
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$[1, x, 2\,x^2-1, 4\,x^3-3\,x, 8\,x^4-8\,x^2+1, 16\,x^5-20\,x^3+5\,x]$ □

Problem 4. (3 XP) Recall that the Bernoulli polynomials $B_n(t)$ are the polynomials characterized by

$$\sum_{n=0}^{\infty} B_n(t)\,\frac{x^n}{n!} = \frac{x\,e^{t\,x}}{e^x-1}.$$

(a) Show that the Bernoulli polynomials satisfy $B_n'(t) = n\,B_{n-1}(t)$.

(b) Further, show that, for $n \geqslant 1$, the Bernoulli polynomials satisfy $\int_0^1 B_n(t)\,\mathrm{d}t = 0$.

(c) Observe that the Bernoulli polynomials are characterized by the initial condition $B_0(t) = 1$ together with the two properties you just showed. Compute the first few Bernoulli polynomials via that route.

(d) Forget that you know the exponential generating function of the Bernoulli polynomials. *Derive* this generating function from the two properties above.

Solution. Let us write $F(x,t) = \frac{x\,e^{t\,x}}{e^x-1}$.

(a) On the level of exponential generating functions, this translates into (review the earlier problem on exponential generating functions)

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x,t) = x\,F(x,t),$$

which is obviously satisfied.

(b) We need to check that

$$\int_0^1 F(x,t)\mathrm{d}t = 1,$$

which is readily done.

(c) For comparison, here are the first few Bernoulli polynomials according to Sage:

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Sage] [bernoulli_polynomial(x,n) for n in [0..5]]
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$$\left[1, x-\frac{1}{2}, x^2-x+\frac{1}{6}, x^3-\frac{3}{2}x^2+\frac{1}{2}x, x^4-2\,x^3+x^2-\frac{1}{30}, x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x\right]$$

(d) The differential equation $\frac{\mathrm{d}}{\mathrm{d}t}F(x,t) = x\,F(x,t)$ implies that $F(x,t)$ is of the form

$$F(x,t) = c(x)e^{x\,t}.$$

Combined with the second property, we then find

$$1 = \int_0^1 F(x,t)\mathrm{d}t = \int_0^1 c(x)e^{x\,t}\mathrm{d}t = \frac{c(x)}{x}(e^x-1),$$

which we solve for $c(x)$ to find $c(x) = x\,/\,(e^x-1)$. In conclusion, as we knew before forgetting, the exponential generating function is

$$F(x,t) = \frac{x\,e^{x\,t}}{e^x-1}.$$ □

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Problem 5. (1 XP) Show that the Bernoulli polynomials have the expansion $B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} t^k$.

Solution. Note the right-hand side is the convolution of two exponential generating functions. Observe that the egf of the sequence $(t^n)_{n \geqslant 0}$ is e^{tx}. Hence, on the level of generating functions, the formula translates to

$$\frac{x e^{tx}}{e^x - 1} = \frac{x}{e^x - 1} \cdot e^{tx},$$

which is obviously true. □

Problem 6. (1 XP) Give a (rough) asymptotic estimate for the Bernoulli numbers B_{2n} as $n \to \infty$.

Solution. The exponential generating function

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}$$

has radius of convergence 2π (the dominant singularity is at $x = 2\pi i$). Hence,

$$\limsup_{n \to \infty} \left(\frac{|B_n|}{n!} \right)^{1/n} = \frac{1}{2\pi}.$$

It follows that, for any $\varepsilon > 0$,

$$\frac{|B_{2n}|}{(2n)!} < \left(\frac{1}{2\pi} + \varepsilon \right)^{2n}$$

for large enough n. It follows from Stirling's approximation that

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n, \quad (2n)! \sim \sqrt{4\pi n} \left(\frac{2n}{e} \right)^{2n},$$

so that, for large enough n,

$$|B_{2n}| < \left(\frac{n}{\pi e} + \varepsilon \right)^{2n}.$$

Indeed, with more effort, one can show that

$$B_{2n} \sim 4(-1)^{n-1} \sqrt{\pi n} \left(\frac{n}{\pi e} \right)^{2n}.$$ □

Problem 7. (2 XP) Let $B_n(x)$ denote the Bernoulli polynomials.

(a) Prove that $1^p + 2^p + \ldots + N^p = \frac{B_{p+1}(N+1) - B_{p+1}(1)}{p+1}$.

(b) Show that $1^3 + 2^3 + \ldots + N^3 = (1 + 2 + \ldots + N)^2$.

Solution.

(a) In class, we showed that

$$\sum_{x=0}^{N-1} x^p = \frac{1}{p+1} \sum_{n=0}^{p} \binom{p+1}{n} B_n N^{p+1-n}.$$

Therefore, using the expansion of the Bernoulli polynomials proved in an earlier exercise, we have

$$\sum_{x=0}^{N-1} x^p = \frac{1}{p+1}\left[\sum_{n=0}^{p+1} \binom{p+1}{n} B_n N^{p+1-n} - B_{p+1}\right] = \frac{B_{p+1}(N) - B_{p+1}(0)}{p+1},$$

which is equivalent to the claimed evaluation

$$\sum_{x=1}^{N-1} x^p = \frac{B_{p+1}(N) - B_{p+1}(1)}{p+1}.$$

Note that both sides only change for $p=0$, because $B_p = B_p(0) = B_p(1)$ with the single exception of $B_1 = B_1(0) = -\frac{1}{2} \neq \frac{1}{2} = B_1(1)$.

Alternatively, computing exponential generating functions directly, we have

$$\sum_{p\geqslant 0} (1^p + 2^p + \ldots + N^p)\frac{x^p}{p!} = \sum_{m=1}^{N} \sum_{p\geqslant 0} m^p \frac{x^p}{p!} = \sum_{m=1}^{N} e^{mx} = e^x \frac{e^{Nx}-1}{e^x-1}$$

on one side, and

$$\sum_{p\geqslant 0} \frac{B_{p+1}(N+1) - B_{p+1}(1)}{p+1} \frac{x^p}{p!} = \frac{1}{x}\sum_{p\geqslant 0} (B_{p+1}(N+1) - B_{p+1}(1)) \frac{x^{p+1}}{(p+1)!} = \frac{1}{x}\left[\frac{x e^{(N+1)x}}{e^x-1} - \frac{x e^x}{e^x-1}\right]$$

on the other sides. Both are clearly equal, and so the identity follows.

(b) Indeed,

$$1^3 + 2^3 + \ldots + N^3 = \frac{B_4(N+1) - B_4(1)}{4} = \left[\frac{N(N+1)}{2}\right]^2 = (1 + 2 + \ldots + N)^2.$$

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Sage] bernoulli_polynomial(x,4)
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$$x^4 - 2\,x^3 + x^2 - \frac{1}{30}$$

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Sage] (bernoulli_polynomial(x+1,4)-bernoulli_polynomial(1,4)).factor()
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$$(x+1)^2\,x^2$$

□

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