Problems #5

Special Functions & WZ Theory
AARMS Summer School, Dalhousie University
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Problem 1. (1 XP) Suppose that the two sequences $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ are equal for large enough n. How is that reflected on their generating functions?

Problem 2. (2 XP) Let $p_M(n)$ be the number of integer partitions of n with parts of size at most M. For instance, $p_2(5)=3$, because we have the partitions $(2,2,1)$, $(2,1,1,1)$, $(1,1,1,1,1)$.

Determine the ordinary generating function $\sum_{n=0}^{\infty} p_M(n)x^n$. Is the sequence $(p_M(n))_{n\geqslant 0}$ C-finite?

Problem 3. Let B_n be the number of partitions of a set of size n. For instance, $B_3=5$ because the set $\{1,2,3\}$ can be partitioned as $\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{1\},\{2\},\{3\}\}$.

(a) **(1 XP)** Express B_{n+1} recursively in terms of $B_n, B_{n-1}, \ldots$

(b) **(1 XP)** Show that the ordinary generating function $F(x)$ of B_n satisfies the functional equation

$$F(x)=1+\frac{x}{1-x}F\left(\frac{x}{1-x}\right).$$

(c) **(1 XP)** Iterate this functional equation to show that we can expand $F(x)$ as

$$F(x)=\sum_{n=0}^{\infty}\frac{x^n}{(1-x)(1-2x)\cdots(1-nx)}.$$

(d) **(1 XP)** Determine the exponential generating function for B_n.

(e) **(1 XP)** Let C_n be the number of partitions of a set of size n such that each part consists of at least 2 elements. For instance, $C_3=4$ because the set $\{1,2,3,4\}$ can be partitioned as $\{\{1,2,3,4\}\}$, $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, $\{\{1,4\},\{2,3\}\}$. Show that $B_n=C_n+C_{n+1}$. Try to give a direct combinatorial proof.

(f) **(1 XP extra)** Determine the exponential generating function for the numbers C_n. Numerically verify your result in Sage.

(g) **(1 XP extra)** Explore the `SetPartitions` command in Sage. For instance:

- Use it to find the 5 partitions of the set $\{1,2,3\}$.
- What is computed by `{x for x in SetPartitions(5) if len(x)<=2}`?
- Similarly, but a little more challenging, what is computed by `{x for x in SetPartitions(5) if min(map(len,x))>=2}`? In particular, what is the interpretation of the following numbers:

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Sage] [len({x for x in SetPartitions(n) if min(map(len,x))>=2}) for n in [1..7]]
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$[0,1,1,4,11,41,162]$

- Explain why `len(SetPartitions(7))` is much slower than `SetPartitions(7).cardinality()`.

Recall that `SetPartitions?` will bring up explanations and examples. Putting a `??` at the end of a function, further shows its source code.

(h) **(1 XP extra)** Experimentally find (i.e. conjecture) the exponential generating function of the number of partitions of a set of size n such that each part consists of at least 3 elements.

(i) **(1 XP extra)** Make a conjecture about the exponential generating function of the number of partitions of a set of size n such that each part consists of at least k elements.

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