Problems #5

Special Functions & WZ Theory

AARMS Summer School, Dalhousie University
Jul 11 – Aug 5, 2016

Problem 1. (1 XP) Suppose that the two sequences $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ are equal for large enough n. How is that reflected on their generating functions?

Solution. If $a_n = b_n$ for $n > N$, then the generating functions differ by a polynomial of degree at most N. □

Problem 2. (2 XP) Let $p_M(n)$ be the number of integer partitions of n with parts of size at most M. For instance, $p_2(5) = 3$, because we have the partitions $(2,2,1)$, $(2,1,1,1)$, $(1,1,1,1,1)$.

Determine the ordinary generating function $\sum_{n=0}^{\infty} p_M(n) x^n$. Is the sequence $(p_M(n))_{n\geqslant 0}$ C-finite?

Solution. By the same argument as for Euler's product formula, we have

$$\sum_{n=0}^{\infty} p_M(n) x^n = \prod_{k=1}^{M} \frac{1}{1-x^k}.$$

Since its generating function is rational, the sequence $(p_M(n))_{n\geqslant 0}$ is C-finite. □

Problem 3. Let B_n be the number of partitions of a set of size n. For instance, $B_3 = 5$ because the set $\{1,2,3\}$ can be partitioned as $\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{1\},\{2\},\{3\}\}$.

(a) **(1 XP)** Express B_{n+1} recursively in terms of $B_n, B_{n-1}, \ldots$

(b) **(1 XP)** Show that the ordinary generating function $F(x)$ of B_n satisfies the functional equation

$$F(x) = 1 + \frac{x}{1-x} F\left(\frac{x}{1-x}\right).$$

(c) **(1 XP)** Iterate this functional equation to show that we can expand $F(x)$ as

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{(1-x)(1-2x)\cdots(1-nx)}.$$

(d) **(1 XP)** Determine the exponential generating function for B_n.

(e) **(1 XP)** Let C_n be the number of partitions of a set of size n such that each part consists of at least 2 elements. For instance, $C_3 = 4$ because the set $\{1,2,3,4\}$ can be partitioned as $\{\{1,2,3,4\}\}$, $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$, $\{\{1,4\},\{2,3\}\}$. Show that $B_n = C_n + C_{n+1}$. Try to give a direct combinatorial proof.

(f) **(1 XP extra)** Determine the exponential generating function for the numbers C_n. Numerically verify your result in Sage.

(g) **(1 XP extra)** Explore the `SetPartitions` command in Sage. For instance:

- Use it to find the 5 partitions of the set $\{1,2,3\}$.
- What is computed by `{x for x in SetPartitions(5) if len(x)<=2}`?
- Similarly, but a little more challenging, what is computed by `{x for x in SetPartitions(5) if min(map(len,x))>=2}`? In particular, what is the interpretation of the following numbers:

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```
Sage] [len({x for x in SetPartitions(n) if min(map(len,x))>=2}) for n in [1..7]]
```

$[0,1,1,4,11,41,162]$

- Explain why `len(SetPartitions(7))` is much slower than `SetPartitions(7).cardinality()`.

  Recall that `SetPartitions?` will bring up explanations and examples. Putting a `??` at the end of a function, further shows its source code.

(h) **(1 XP extra)** Experimentally find (i.e. conjecture) the exponential generating function of the number of partitions of a set of size n such that each part consists of at least 3 elements.

(i) **(1 XP extra)** Make a conjecture about the exponential generating function of the number of partitions of a set of size n such that each part consists of at least k elements.

Solution. The numbers B_n are known as Bell numbers, but go back to at least medieval Japan.

(a) By definition, B_{n+1} is the number of distinct partitions of the set $\{0,1,2,\ldots,n\}$. Denote with $\mathcal{P}$ such a partition. Let us focus on the element 0, and let $P_0\in\mathcal{P}$ be the set containing 0. The size $k=|P_0|$ of that set can be anything from $k=1$ to $k=n+1$.

Given that size k, there are a total of $\binom{n}{k-1}$ many possibilities for the set P_0.

Given P_0, there are B_{n+1-k} many ways to partition the remaining $n+1-k$ many elements.

In other words,

$$B_{n+1}=\sum_{k=1}^{n+1}\binom{n}{k-1}B_{n+1-k}=\sum_{k=0}^{n}\binom{n}{k}B_k. \tag{1}$$

(b) By a previous exercise, we know that, for any sequence B_n, if $F(x)$ is the ordinary generating of B_n, then

$$\sum_{n\geqslant 0}\left(\sum_{k=0}^{n}\binom{n}{k}B_k\right)x^n=\frac{1}{1-x}F\left(\frac{x}{1-x}\right).$$

Taking the generating function of both sides of (1), we therefore obtain

$$\frac{F(x)-1}{x}=\frac{1}{1-x}F\left(\frac{x}{1-x}\right),$$

which implies the claimed functional equation.

As noted in [Kla03], the substitution $x\to x/(1+x)$ results in the equivalent functional equation

$$F\left(\frac{x}{1+x}\right)=1+xF(x).$$

By the way, it is shown in [Kla03] that the ordinary generating function $F(x)$ of the Bell numbers does not satisfy any algebraic differential equation. The proof shows that any such algebraic differential equation would be incompatible with the functional equation we just derived. However, we will see below that the exponential generating function can be simply expressed in terms of known functions.

(c) Let us do one iteration of the functional equation to (hopefully) spot patterns:

$$F(x)=1+\frac{x}{1-x}F\left(\frac{x}{1-x}\right)=1+\frac{x}{1-x}\left[1+\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}F\left(\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}\right)\right]=1+\frac{x}{1-x}\left[1+\frac{x}{1-2x}F\left(\frac{x}{1-2x}\right)\right]$$

You surely can guess what happens in the next iteration: we expect that the argument $\frac{x}{1-2x}$ will get replaced with $\frac{x}{1-3x}$. Indeed, for any n, we have

$$\frac{\frac{x}{1-x}}{1-n\frac{x}{1-x}}=\frac{x}{1-(n+1)x}.$$

What we found is that

$$F(x)=1+\frac{x}{1-x}\left(1+\frac{x}{1-2x}\left(1+\frac{x}{1-3x}(1+\ldots)\right)\right).$$

It is important to realize that the factors $\frac{x}{1-nx}$ are of the form $x+O(x^2)$, and so multiplying with them increases the degrees of x. Multiplying the iterated expression out, we therefore obtain

$$F(x)=\sum_{n=0}^{\infty}\frac{x^n}{(1-x)(1-2x)\cdots(1-nx)}.$$

Note that we can avoid any question of convergence during the argument since we are working with a formal power series.

(d) The recurrence implies that $G'(x)=e^xG(x)$. Hence, $G(x)=Ce^{e^x}$. Since $G(0)=1$, we find $G(x)=e^{e^x-1}$.

(e) The partitions of $\{1,2,\ldots,n\}$ that are counted by B_n come in two flavors:

- The ones where each part has size at least 2, and these are counted by C_n.
- Those that contain some parts $\{x_1\}, \{x_2\}, \ldots$ of size 1. We need to show that there are C_{n+1} many of these. So, here we are comparing with partitions of $\{0,1,2,\ldots,n\}$, where we gained an extra element 0. The crucial idea is almost jumping at us: we should do something with this new element as well as with the singletons $\{x_1\}, \{x_2\}, \ldots$, and the most natural thing to do is to throw them all together as a part. Keeping the other parts untouched, we now have a partition $\{0,1,2,\ldots,n\}$ whose parts have size at least 2. Indeed, this way, we obtain any such partition exactly once, and there are C_{n+1} many of them.

Combined, we therefore have $B_n=C_n+C_{n+1}$.

(f) The identity $B_n=C_n+C_{n+1}$ translates into

$$e^{e^x-1}=H(x)+H'(x).$$

This is a linear differential equation, which we can therefore solve with an integrating factor. Here, we multiply both sides with e^x so that

$$e^xe^{e^x-1}=e^xH(x)+e^xH'(x)=\frac{\mathrm{d}}{\mathrm{d}x}(e^xH(x)).$$

We can now integrate both sides to find

$$e^xH(x)=e^{e^x-1}+C.$$

Since $H(0)=C_0=1$, we have $C=0$, and hence

$$H(x)=e^{e^x-1-x}.$$

We compute the first few terms of the Taylor series to verify:

```
Sage] egf = exp(exp(x)-1-x).series(x, 9)
Sage] egf
```

$$1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{6}x^4+\frac{11}{120}x^5+\frac{41}{720}x^6+\frac{9}{280}x^7+\frac{143}{8064}x^8+\mathcal{O}(x^9)$$

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```
Sage] C = [egf.coefficient(x,n)*factorial(n) for n in [0..8]]
Sage] C
```

$[1, 0, 1, 1, 4, 11, 41, 162, 715]$

```
Sage] [C[n]+C[n+1] for n in [0..7]]
```

$[1, 1, 2, 5, 15, 52, 203, 877]$

```
Sage] [bell_number(n) for n in [0..7]]
```

$[1, 1, 2, 5, 15, 52, 203, 877]$

```
Sage]
```

(g) Explore!

```
Sage] list(SetPartitions({1,2,3}))
```

$[\{\{1, 2, 3\}\}, \{\{1\}, \{2, 3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1, 2\}, \{3\}\}, \{\{1\}, \{2\}, \{3\}\}]$

```
Sage] [x for x in SetPartitions(5) if len(x)<=2]
```

$[\{\{1, 2, 3, 4, 5\}\}, \{\{1\}, \{2, 3, 4, 5\}\}, \{\{1, 3, 4, 5\}, \{2\}\}, \{\{1, 2, 4, 5\}, \{3\}\}, \{\{1, 2, 3, 5\}, \{4\}\}, \{\{1, 2, 3, 4\}, \{5\}\}, \{\{1, 2\}, \{3, 4, 5\}\}, \{\{1, 3\}, \{2, 4, 5\}\}, \{\{1, 4\}, \{2, 3, 5\}\}, \{\{1, 5\}, \{2, 3, 4\}\}, \{\{1, 4, 5\}, \{2, 3\}\}, \{\{1, 3, 5\}, \{2, 4\}\}, \{\{1, 3, 4\}, \{2, 5\}\}, \{\{1, 2, 5\}, \{3, 4\}\}, \{\{1, 2, 4\}, \{3, 5\}\}, \{\{1, 2, 3\}, \{4, 5\}\}]$

These are all the set partitions of $\{1, 2, 3, 4, 5\}$ into at most two parts.

```
Sage] [len({x for x in SetPartitions(n) if min(map(len,x))>=2}) for n in [1..7]]
```

$[0, 1, 1, 4, 11, 41, 162]$

This is the number C_n of partitions of a set of size n such that each part consists of at least 2 elements.

The reason why `len(SetPartitions(7))` is much slower than `SetPartitions(7).cardinality()` is that `SetPartitions(7).cardinality()` actually calls `bell_number(7)` and never constructs the 877 set partitions.

(h) The conjecture making is left for the thrill of it.

□

Bibliography

[Kla03] Martin Klazar. Bell numbers, their relatives, and algebraic differential equations. *Journal of Combinatorial Theory, Series A*, 102(1):63–87, April 2003.