Problems #4

Problem 1. (1 XP) Prove that $2(-4)^n \binom{1/2}{n+1} = \frac{1}{n+1}\binom{2n}{n}$.

Problem 2. (2 XP) Let C_n be the nth Catalan number.

(a) Show that C_n counts the number of "legal" expressions that can be formed using n pairs of parentheses. For instance, $C_3 = 5$ because we have the possibilities ((())), (()()), (())(), ()(()), ()()().

(b) **(bonus; 2 XP extra)** Show that C_n also counts the number of permutations of $\{1, 2, ..., n\}$ that are 123-avoiding. That is, those permutations $\pi_1\pi_2...\pi_n$ such that we do not have $i < j < k$ with $\pi_i < \pi_j < \pi_k$.

For instance, the 123-avoiding permutations of $\{1, 2, 3, 4\}$ are the $C_4 = 14$ permutations 1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312, 4321. On the other hand, 2314 is not 123-avoiding because it contains 234 as a substring.

Exploring Sage

Problem 3. (2 XP extra) Explore `CFiniteSequences` in Sage.

It turns out that C-finite sequences are closed under the Hadamard product, that is, if a_n and b_n are Cfinite, then the product $c_n = a_n b_n$ is C-finite. Unfortunately, this closure property is not yet implemented in Sage. Nevertheless, find a (possibly heuristic) way to find the generating function of F_n^2, the square of the Fibonacci numbers.

Problem 4. (2 XP extra) Jeff Lagarias proved in 2002 that the Riemann hypothesis is equivalent to

$$\sigma(n) < H_n + \ln(H_n) e^{H_n}$$

for all $n > 1$. Here, $\sigma(n) = \sum_{d|n} d$ is the sum of the divisors of n. Obtain numerical evidence using Sage by verifying that the inequality holds for small n. Also, make plots to get a visual impression.

Problem 5. (1 XP extra) Define the following function $A(n)$ in Sage:

```
Sage] def A(n):
          if n==0: return 1
          return 2*(2*n-1)/(n+1) * A(n-1)
Sage] [ A(n) for n in [0..10] ]
```

$[1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796]$

(a) Show that $A(n)$ equals the n-th Catalan number, that is, $A(n) = \frac{1}{n+1}\binom{2n}{n}$.

(b) Show that $A(n) = \binom{2n}{n} - \binom{2n}{n+1}$.

(c) Observe that `A(1).parent()` is the rational numbers, even though 1 is an integer. This is the result of using the division operator `/`. Use the operator `//` to rewrite the function $A(n)$ so that its output is always an integer.

Armin Straub
straub@southalabama.edu