Problems #2

Problem 1. (warmup, 2 XP) Suppose that the sequence $(a_n)_{n\geqslant 0}$ has ordinary generating function $F(x)$. For each of the following choices of b_n, express the generating function of $(b_n)_{n\geqslant 0}$ in terms of $F(x)$.

(a) $b_n = a_{n+3}$

(b) $b_n = n^2 a_n$

(c) $b_n = (n+1)(a_n - 2)$

(d) $b_n = a_{2n}$

(e) $b_n = \sum_{k=0}^{n} (-1)^k a_k$

Problem 2. (warmup, 1 XP) Show that $\sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}$. [Use binomial and/or geometric series!]

Problem 3. (2 XP) Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ be the harmonic numbers. Show that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$.

Problem 4. (2 XP) Suppose that the sequence $(a_n)_{n\geqslant 0}$ has ordinary generating function $F(x)$.

(a) Express the ordinary generating function for $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$ in terms of $F(x)$.

(b) The *binomial transform* of a sequence a_n is the sequence $b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k$. What is the binomial transform of the binomial transform of a sequence?

Problem 5. (3 XP) The exponential generating function of a sequence $(a_n)_{n\geqslant 0}$ is the (formal) power series $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

Suppose that the sequences $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ have exponential generating functions $F(x)$ and $G(x)$.

(a) Which sequence is generated by $F'(x)$? By $xF(x)$? By $F(x)G(x)$?

(b) What is the exponential generating function of $n a_n$? Of $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$?

(c) What is the exponential generating function of the binomial transform of a_n? Revisit the question what the binomial transform of the binomial transform of a sequence is.

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