Example 157. Consider the polygonal region with vertices $(0,0)$, $(4,0)$, $(4,2)$, $(2,2)$, $(2,3)$, $(0,3)$. We wish to find the steady-state temperature distribution $u(x,y)$ within this region if the temperature is A between $(0,0)$ and $(4,0)$, and B elsewhere on the boundary.

Spell out the resulting equations when we discretize this problem using a step size of $h=1$.

Solution. As before, we write $u_{m,n}=u(mh,nh)$. Make a sketch!

$$\begin{array}{ccccc} & B & & & \\ B & u_{1,2} & B & B & \\ B & u_{1,1} & u_{2,1} & u_{3,1} & B \\ & A & A & A & \end{array}$$

If we approximate $u_{xx}+u_{yy}$ by $\frac{1}{h^2}[u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx}+u_{yy}=0$ translates into

$$u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1}-4u_{m,n}=0.$$

Spelling out these equation in matrix-vector form, we obtain:

$$\begin{bmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 1 & 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \end{bmatrix}=\begin{bmatrix} -A-B \\ -A-B \\ -A-2B \\ -3B \end{bmatrix}$$

Comment. Note that, because of the way we discretize, it matters that there is a well-defined temperature at the boundary vertex $(2,2)$. For the other vertices, we don't need a well-defined temperature (and so it is not a problem that it is unclear what the temperature should be at $(0,0)$ or $(4,0)$ where it jumps from A to B).

A glance at the heat equation

We wish to describe one-dimensional heat flow.

Comment. If this sounds very specialized, it might help to know that the heat equation is also used, for instance, in probability (Brownian motion), financial math (Black-Scholes), or chemical processes (diffusion equation).

Let $u(x,t)$ describe the temperature at time t at position x.

If we model a heated rod of length L, then $x\in[0,L]$.

(heat equation)

$$u_t=ku_{xx}$$

Here $k>0$ is a constant (the thermal diffusivity).

How to see that this PDE describes heat flow? Experience tells us that heat flows from warmer to cooler areas and has an averaging effect.

Make a sketch of some temperature profile $u(x,t)$ for fixed t.

As t increases, we expect maxima (where $u_{xx}<0$) of that profile to flatten out (which means that $u_t<0$); similarly, minima (where $u_{xx}>0$) should go up (meaning that $u_t>0$). The simplest relationship between u_t and u_{xx} which conforms with our expectation is $u_t=ku_{xx}$, with $k>0$.

Higher dimensions. In higher dimensions, the heat equation takes the form $u_t=k(u_{xx}+u_{yy})$ or $u_t=k(u_{xx}+u_{yy}+u_{zz})$. When the temperature has stabilized (so that $u_t=0$), we obtain the Laplace equation (for steady-state heat distribution).

Armin Straub
straub@southalabama.edu

Let us think about what is needed to describe a unique solution of the heat equation.

- **Initial condition** at $t=0$: $u(x,0)=f(x)$ (IC)

  This specifies an initial temperature distribution at time $t=0$.

- **Boundary condition** at $x=0$ and $x=L$: (BC)

  Assuming that heat only enters/exits at the boundary (think of our rod as being insulated, except possibly at the two ends), we need some condition on the temperature at the ends. For instance:

  - $u(0,t)=A$, $u(L,t)=B$

    This models a rod where one end is kept at temperature A and the other end at temperature B.

  - $u_x(0,t)=u_x(L,t)=0$

    This models a rod whose ends are insulated as well.

Under such assumptions, our physical intuition suggests that there should be a unique solution.

Discretizing the 1D heat equation

In the following we discuss one approach to discretizing the 1D heat equation (known as "forward-time central-space"). As for the Laplace equation, we use the central difference to approximate the second derivative (with respect to the spatial variable x) as

$$u_{xx}(x,t)\approx\frac{1}{h^2}[u(x+h,t)-2u(x,t)+u(x-h,t)].$$

On the other hand, to discretize the time derivative u_t we use the forward difference approximation

$$u_t(x,t)\approx\frac{1}{\delta}[u(x,t+\delta)-u(x,t)].$$

Here, h represents the step size in space (x), and δ represents the step size in time (t).
We then approximate the 1D heat equation $u_t=ku_{xx}$ by

$$\frac{1}{\delta}[u(x,t+\delta)-u(x,t)]=\frac{k}{h^2}[u(x+h,t)-2u(x,t)+u(x-h,t)].$$

If we write $u_{m,j}=u(mh,j\delta)$ for the temperature at lattice point m and time step j then this can be written as

$$u_{m,j+1}-u_{m,j}=\frac{k\delta}{h^2}[u_{m+1,j}-2u_{m,j}+u_{m-1,j}]$$

Note that the initial condition $u(x,0)=f(x)$ determines $u_{m,j}$ for $j=0$. We therefore solve the above equation for $u_{m,j+1}$ (this is the value at the next time step) to get

$$u_{m,j+1}=\sigma u_{m+1,j}+(1-2\sigma)u_{m,j}+\sigma u_{m-1,j},\quad \sigma=\frac{k\delta}{h^2}.$$

This explicit formula allows us to compute the future state based entirely on the current state.

Advanced comment. Unlike the Laplace discretization, this approach is only stable if the time step is small enough compared to the spatial step size. Specifically, we require $\delta\leqslant\frac{h^2}{2k}$ (so that $\sigma\leqslant\frac{1}{2}$).

Example 158. Consider the heat flow problem:

$$u_t = \frac{1}{3}u_{xx} \quad \text{(PDE)}$$
$$u(0,t)=1, \quad u(1,t)=2 \quad \text{(BC)}$$
$$u(x,0)=x(1-x), \quad x\in(0,1) \quad \text{(IC)}$$

(a) Discretize using a spatial step of $h=\frac{1}{4}$ and a time step of $\delta=\frac{1}{32}$.

(b) Determine the approximated values after the first two time steps.

(c) Determine the exact steady-state temperature distribution.

Solution.

(a) Since $x\in[0,1]$ and $t\geqslant 0$, we have values $u_{m,j}=u(mh,j\delta)$ where $m\in\{0,1,2,3,4\}$ and $j\in\{0,1,2,\ldots\}$.
The boundary condition (BC) implies that $u_{0,j}=1$ and $u_{4,j}=2$ for all $j\geqslant 0$.
The initial condition (IC) determines the values $u_{m,j}$ when $j=0$. Specifically,

$$\begin{bmatrix} u_{1,0} \\ u_{2,0} \\ u_{3,0} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\left(1-\frac{1}{4}\right) \\ \frac{1}{2}\left(1-\frac{1}{2}\right) \\ \frac{3}{4}\left(1-\frac{3}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{3}{16} \\ \frac{1}{4} \\ \frac{3}{16} \end{bmatrix}.$$

Here, $\sigma=\frac{k\delta}{h^2}=\frac{1}{6}$ and our discretization above becomes

$$u_{m,j+1}=\frac{1}{6}u_{m+1,j}+\frac{2}{3}u_{m,j}+\frac{1}{6}u_{m-1,j}.$$

In matrix-vector form, this takes the form (we use $u_{0,j}=1$ and $u_{4,j}=2$)

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} \\ 0 \\ \frac{1}{3} \end{bmatrix}.$$

(b) We find

$$\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{3}{16} \\ \frac{1}{4} \\ \frac{3}{16} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} \\ 0 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{11}{48} \\ \frac{1}{2} \end{bmatrix} \approx \begin{bmatrix} 0.3333 \\ 0.2292 \\ 0.5000 \end{bmatrix}$$

as well as

$$\begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{11}{48} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} \\ 0 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{41}{96} \\ \frac{7}{24} \\ \frac{203}{288} \end{bmatrix} \approx \begin{bmatrix} 0.4271 \\ 0.2917 \\ 0.7049 \end{bmatrix}.$$

(c) Using $u_t=0$, we find that the steady-state heat distribution $u(x)$ satisfies $\frac{1}{3}u''(x)=0$ as well as $u(0)=1$, $u(1)=2$ (these follow from the boundary condition).
The solutions to $u''(x)=0$ consist of linear functions $u(x)=Ax+B$. Using the two boundary conditions, we find $A=1$ and $B=1$. Hence, $u(x)=x+1$.

Comment. Note that this fits well with our intuitive understanding that the (stabilized) temperature inside a rod, where one end is held at temperature 1 and the other at temperature 2, should interpolate between those two values (and that's what $u(x) = x + 1$ does in a linear way).

For comparison. After 32, 64 and 96 time steps (that is, $t = 1, 2, 3$), our approximations are

$$\begin{bmatrix} u_{1,32} \\ u_{2,32} \\ u_{3,32} \end{bmatrix} = \begin{bmatrix} 1.2090 \\ 1.4420 \\ 1.7090 \end{bmatrix}, \quad \begin{bmatrix} u_{1,64} \\ u_{2,64} \\ u_{3,64} \end{bmatrix} = \begin{bmatrix} 1.2485 \\ 1.4978 \\ 1.7485 \end{bmatrix}, \quad \begin{bmatrix} u_{1,96} \\ u_{2,96} \\ u_{3,96} \end{bmatrix} = \begin{bmatrix} 1.2499 \\ 1.4999 \\ 1.7499 \end{bmatrix}.$$

These are approaching the true steady-state values $\begin{bmatrix} u(1/4) \\ u(1/2) \\ u(3/4) \end{bmatrix} = \begin{bmatrix} 5/4 \\ 3/2 \\ 7/4 \end{bmatrix}$.