As an example, we apply repeated Richardson extrapolation to Euler's method (which is order $1$) until we have an approximation of order $4$. While this is comparable to RK4, we observe that the latter has less numerical issues.

**Example 152.** Suppose that $A(h)$ is an approximation of $A^*$ of order $1$.

(a) Combine $A(h)$ and $A(h/2)$ to get a new approximation $B(h)$ of order $2$ (or higher).

(b) Combine $B(h)$ and $B(h/2)$ to get a new approximation $C(h)$ of order $3$ (or higher).

(c) Spell out the approximation $C(h)$ in terms of the original approximation.

(d) By continuing the above, produce an order $4$ approximation.

**Solution.**

(a) Since $A(h)$ is of order $1$, we have $A(h) = A^* + Ch + O(h^2)$ as well as $A\left(\frac{h}{2}\right) = A^* + \frac{1}{2}Ch + O(h^2)$.

It follows that $\frac{2A\left(\frac{h}{2}\right) - A(h)}{2-1} = A^* + O(h^2)$. This is just Richardson extrapolation.

In conclusion, $B(h) = 2A\left(\frac{h}{2}\right) - A(h)$ is an approximation of $A^*$ of order (at least) $2$.

(b) We assume that $B(h) = A^* + Ch^2 + O(h^3)$ (for a new constant $C$). Hence, $B\left(\frac{h}{2}\right) = A^* + \frac{1}{4}Ch + O(h^3)$.

It follows that $C(h) := \frac{4B\left(\frac{h}{2}\right) - B(h)}{4-1} = A^* + O(h^3)$. Again, this is just another Richardson extrapolation.

(c) We have:

$$\begin{aligned} C(h) &= \frac{4}{3}B\left(\frac{h}{2}\right) - \frac{1}{3}B(h) = \frac{4}{3}\left(2A\left(\frac{h}{4}\right) - A\left(\frac{h}{2}\right)\right) - \frac{1}{3}\left(2A\left(\frac{h}{2}\right) - A(h)\right) \\ &= \frac{8}{3}A\left(\frac{h}{4}\right) - 2A\left(\frac{h}{2}\right) + \frac{1}{3}A(h) \end{aligned}$$

(d) Since $C(h)$ is an order $3$ approximation, the following is an order $4$ (or higher) approximation:

$$\frac{8C\left(\frac{h}{2}\right) - C(h)}{8-1} = \frac{64}{21}A\left(\frac{h}{8}\right) - \frac{8}{3}A\left(\frac{h}{4}\right) + \frac{2}{3}A\left(\frac{h}{2}\right) + \frac{1}{21}A(h)$$

**Comment.** Instead of going via the intermediate approximation $B(h)$, we could have also proceeded as in the first part of the next example: starting with $A(h) = A^* + C_1h + C_2h^2 + O(h^3)$, we would look for a combination $\alpha A(h) + \beta A\left(\frac{h}{2}\right) + \gamma A\left(\frac{h}{4}\right)$ such that $C_1$ and $C_2$ get eliminated. This alternative approach is probably a bit more tedious to spell out but has the advantage that we can combine any three approximations.

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**Example 153.** Python Let us use the Euler method as implemented in Example 142. We apply it to the IVP $y' = \frac{2}{5}y^2$ with $y(0) = 2$, which has the exact solution $y(x) = \frac{10}{5 - 4x}$ (note that this has a singularity at $x = \frac{5}{4}$) with $y(1) = 10$.

```
>>> def f(x, y):
        return 2/5*y**2
```

The following confirms that the error in Euler's method is $O(h)$.

```
>>> [euler(f, 0, 2, 1, 10**n)[-1] - 10 for n in range(7)]

  [-6.4, -3.153039180044801, -0.5786318726117212, -0.06365106772819829,
  -0.006430401713281242, -0.0006437015789977352, -6.437677910753337e-05]
```

If $A(h)$ is Euler (which is order $1$) then $B(h) = 2A\left(\frac{h}{2}\right) - A(h)$ is an approximation of order $2$:

```
>>> [2*euler(f, 0, 2, 1, 2*10**n)[-1]-euler(f, 0, 2, 1, 10**n)[-1] - 10 for n in range(7)]

  [-4.864000000000001, -1.0300367298617736, -0.03059704698782717,
  -0.00036086415725833376, -3.672563401835305e-06, -3.678901450143712e-08,
  -3.710773910370335e-10]
```

And we know how to further increase the order. The following is order $3$:

```
>>> def A(n):
        return euler(f, 0, 2, 1, n)[-1]

>>> [8/3*A(4*10**n)-2*A(2*10**n)+1/3*A(10**n) - 10 for n in range(7)]

  [-3.261606554763264, -0.23097082059404528, -0.0009539131127027645,
  -1.1759506524100516e-06, -1.2013980921210532e-09, -4.835243316847482e-12,
  2.644640062499093e-11]
```

Observe that the last approximation reveals numerical issues. These worsen if we push things to order $4$:

```
>>> [64/21*A(8*10**n)-8/3*A(4*10**n)+2/3*A(2*10**n)-1/21*A(10**n) - 10 for n in range(7)]

  [-1.7873721196484613, -0.03201840214106966, -1.63900245677695e-05,
  -2.0771775410821647e-09, 1.0391687510491465e-12, 5.448086426440568e-12,
  -2.8705926524708048e-11]
```

For comparison, let us also apply RK4 which is of order $4$ as well:

```
>>> [runge_kutta4(f, 0, 2, 1, 10**n)[-1] - 10 for n in range(5)]

  [-1.6372498407533236, -0.0029737693332965165, -3.5183410496131273e-07,
  -3.524114333686157e-11, -6.394884621840902e-14]
```

Here, the numerical issue with the final approximation is less pronounced.

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# A glance at discretizing PDEs

One of the most important partial differential equations is the following equation which, for instance, models the steady-state temperature $u(x, y)$ of a region in two-dimensional space.

**(Laplace equation, 2D)**

$$u_{xx} + u_{yy} = 0$$

**Notation.** Here, for instance, $u_{xx} = \frac{\partial^2}{\partial x^2} u(x, y)$ is used to denote two partial derivatives with respect to $x$.

The Laplace equation is often written as $\Delta u = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator you may know from Calculus III.

The Laplacian $\Delta u$ is also often written as $\Delta u = \nabla^2 u$. The operator $\nabla = (\partial / \partial x, \partial / \partial y)$ is pronounced "nabla" (Greek for a certain harp) or "del" (Persian for heart), and $\nabla^2$ is short for the inner product $\nabla \cdot \nabla$.

**Higher dimensions.** Likewise, the Laplace equation in three dimensions is $\Delta u = 0$ with $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

**Comment.** The Laplace equation is so important that its solutions have their own name: **harmonic functions**.

It is also known as the **potential equation** and is satisfied by electric/gravitational potential functions.

Recall from Calculus III (if you have taken that class) that the gradient of a scalar function $f(x, y)$ is the vector field $\boldsymbol{F} = \operatorname{grad} f = \nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$. One says that $\boldsymbol{F}$ is a **gradient field** and $f$ is a **potential function** for $\boldsymbol{F}$ (for instance, $\boldsymbol{F}$ could be a gravitational field with gravitational potential $f$).

One way to describe a unique solution to the Laplace equation within a region is by specifying the values of $u(x, y)$ along the boundary of that region.

This is particularly natural for steady-state temperatures profiles of a region $R$. The Laplace equation governs how temperature behaves inside the region but we need to also prescribe the temperature on the boundary.

The PDE with such a boundary condition is called a Dirichlet problem:

**(Dirichlet problem)**

$u_{xx} + u_{yy} = 0$ within region $R$

$u(x, y) = f(x, y)$ on boundary of $R$

**In general.** A Dirichlet problem consists of a PDE, that needs to hold within a region $R$, and prescribed values on the boundary of that region ("Dirichlet boundary conditions").

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